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Bi-decomposition of multi-valued logical functions and its applications[☆]

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ABSTRACT

The bi-decomposition of multi-valued logical (MVL) functions, including disjoint and non-disjoint cases, is considered. Using a semi-tensor product, an MVL function can be expressed in its algebraic form. Based on this form, straightforward verifiable necessary and sufficient conditions are provided for each case, respectively. The constructive proofs also lead to constructing corresponding decompositions. Using these results, the implicit function theorem (IFT) of k-valued functions, as a special bi-decomposition, is obtained. Finally, as an application, the normalization of dynamic–algebraic (D–A) Boolean networks is investigated using IFT of k-valued functions.

1. Introduction

Denote by $\mathcal{D}_k := \left\{0, \frac{1}{k-1}, \frac{2}{k-1}, \dots, \frac{k-2}{k-1}, 1\right\}$ the domain of a k-valued logical variable (for convenience, $\mathcal{D} := \mathcal{D}_2$). Let $x_i \in \mathcal{D}_{k_i}$, $i = 1, \dots, n$, be k_i-valued logical variables:

$$f(x_1, \dots, x_n) : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0} \tag{1}$$

is called a multi-valued logical (MVL) function (see Fig. 1(a)). When $k_i = k, i = 0, 1, \dots, n$, it is called a k-valued logical function. In addition, if $k = 2$ it becomes a Boolean function.

MVL functions have been investigated and applied to various research areas such as data mining (Files & Perkowski, 1998; Zupan, Bohanec, Demsar, & Bratko, 1998), game theory (Zhao, Li, & Cheng, 2011), and circuit theory (Hachtel & Somenzi, 2002). For instance, digital circuits that are realized by MVL functions have some advantages over the dominatingly used Boolean functions (Miller & Thornton, 2007). Moreover, although MVL functions can be merged into Boolean functions by proper encoding (Didier, Remy, & Chaouiya, 2011; Mishchenko & Brayton, 2002), unnecessary complexity would be brought in along the way and fundamental difficulties still exist (Lang & Steinbach, 2003). Therefore, the study of the properties of MVL functions has its own importance.

Let $\{X_1, X_2\}$ be a partition of $X = \{x_1, x_2, \dots, x_n\}$. If f can be expressed as

$$f(X) = F(\phi(X_1), \psi(X_2)), \tag{2}$$

then f can be realized in a disjoint bi-decomposed form as in Fig. 1(b). Let $\{X_1, X_2, X_3\}$ be a partition of X. If f can be expressed as

$$f(X) = F(\phi(X_1, X_2), \psi(X_2, X_3)), \tag{3}$$

then f can be realized in a non-disjoint bi-decomposed form as in Fig. 1(c). Here, F, ϕ, ψ are all MVL functions, where ϕ, ψ are called the decomposition functions, and F is called the operator function or a *gate*. Throughout this paper we assume $\phi(\cdot), \psi(\cdot) \in \mathcal{D}_{k_0}$, and therefore F is called a k_0-gate to emphasize the output cardinality.

The decomposition of MVL functions is a useful technique because it can reduce the complexity of the problem involved and reveal the intrinsic structure of the functions. In circuit theory, the decomposition of logical (switching) functions can significantly improve circuit performance in terms of chip area, operation speed and power consumption. Therefore, it is a long standing research topic since 1950s, and there are some interesting and useful results. For Boolean functions, when the number of inputs of a switching circuit is small, the Quine–McCluskey procedure was widely used for designing a two-stage network (Davio, Deschamps, & Thayse, 1978), while when a large number of inputs is involved,

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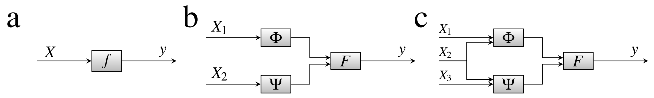

Fig. 1. (a) MVL function, (b) disjoint decomposition, (c) and non-disjoint decomposition.

the decomposition chart method to multi-level minimization was proposed by Ashenhurst (Ashenhurst, 1957) and was further discussed by Curtis (Curtis, 1962), with related efficient algorithms developed later by many other authors (Karp, 1963; Mishchenko, Steinbach, & Perkowski, 2001; Posthoff & Steinbach, 2004; Sasao, 1999). As for MVL functions, Hurst (1984) gave an overview of MVL functions, Lang and Steinbach (2003) discussed the bi-decomposition for the min- and max-operators through MVL differential calculus, and we refer the reader to Brayton and Khatri (1999) and Vykhovanets (2006) and references therein for more details.

This paper focuses on the theoretical aspect of bi-decomposition of MVL functions and its byproduct—the global implicit function theorem (IFT) and its application to the normalization of Boolean networks. In this paper, we first provide easily verifiable necessary and sufficient conditions for both disjoint and non-disjoint bi-decompositions of MVL functions. Using the bi-decomposition result, a global IFT for k-valued functions is obtained, which can be seen as a special kind of bi-decomposition. Finally, the above results are used to convert a dynamic–algebraic (D–A) Boolean (control) network into its standard form, which makes the tools and results developed for MVL (control) networks applicable to D–A Boolean (control) networks.

The basic tool for our approach is the semi-tensor product (STP) of matrices and the matrix expression of logic (Cheng, Qi, & Zhao, 2012). It has been successfully applied to the analysis of topological structure of Boolean as well as general logical networks (Cheng, 2009; Cheng & Qi, 2010a) and the synthesis of logical control networks (Cheng, 2011; Cheng & Qi, 2009, 2010b; Cheng, Qi, & Li, 2011; Laschov & Margaliot, 2011; Li & Sun, 2011). The key technique of this approach is converting an MVL function into its algebraic form, which allows the application of matrix and analyzing tools for discrete-time systems to MVL dynamic systems.

The rest of this paper is organized as follows. Section 2 is a survey for STP and the matrix expression of MVL functions. Only limited concepts and results, which are used in the sequel, are introduced. Section 3 considers the disjoint bi-decomposition of MVL functions. Non-disjoint bi-decomposition of MVL functions is discussed in Section 4. Necessary and sufficient conditions are presented for each case. Section 5 presents a global IFT for k-valued functions which is considered as a special bi-decomposition. In Section 6, the IFT is used to converting a D–A Boolean network into its normal form. Section 7 consists of some concluding remarks.

2. Matrix expression of logic

For statement convenience, we first introduce some notations.

- $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
- $Col_i(A)$: the i-th column of matrix A.
- $\delta_n^k := Col_k(I_n)$ where I_n is the $n \times n$ identity matrix.
- $\Delta_n := \{\delta_n^1, \dots, \delta_n^n\}$; $\Delta := \Delta_2$.
- A matrix $A \in \mathcal{M}_{n\times m}$ is called a logical matrix if its columns $\mathrm{Col}(A) \subset \Delta_n$. The set of $n \times m$ logical matrices is denoted by $\mathcal{L}_{n\times m}$.
- Let $A \in \mathcal{L}_{n\times m}$. Then it is compactly expressed as
  $$A = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_m}] := \delta_n[i_1, i_2, \dots, i_m].$$
- $A = diag(A_1, \dots, A_r)$ is the block diagonal matrix of which the i-th diagonal block is A_i.

Table 1
Table of f.

(x, y)	(1, 1)	(1, 0)	(0.5, 1)	(0.5, 0)	(0, 1)	(0, 0)
z	0.5	1	0.5	0	1	0.5

Definition 1 (*Cheng & Qi, 2010b*). Let $M \in \mathcal{M}_{m\times n}$ and $N \in \mathcal{M}_{p\times q}$. The semi-tensor product of matrices, denoted by $M \ltimes N$, is defined as

$$M \ltimes N := \left(M \otimes I_{s/n}\right)\left(N \otimes I_{s/p}\right), \tag{4}$$

where $s = \mathrm{lcm}\{n, p\}$ is the least common multiple of n and p; $\otimes$ is the Kronecker product of matrices.

Remark 2. When $n = p$, STP coincides with a conventional matrix product. So STP is a generalization of a conventional matrix product, and all the properties of the conventional matrix product (such as the associativity and distributivity) remain correct.

Throughout this paper the matrix product is assumed to be STP. In the sequel, the symbol $\ltimes$ is omitted when there is no confusion.

Some new properties of STP, which are used in the sequel, are given here for convenience.

Proposition 3 (*Cheng et al., 2011*).

(i) *Let $x \in \mathbb{R}^t$ be a column vector and A a given matrix. Then*
$$xA = (I_t \otimes A)x. \tag{5}$$

(ii) *Let $x \in \Delta_k$, $R_k := \delta_{k^2}[1, k+2, \dots, (i-1)k+i, \dots, k^2] \in \mathcal{L}_{k^2\times k}$. Then*
$$x^2 = R_k x. \tag{6}$$

Next, for any integer k ($k \geq 2$), we identify $\frac{i}{k-1} \sim \delta_k^{k-i}$, $i = 0, 1, \dots, k-1$. Then $x \in \mathcal{D}_k$ has its corresponding vector form (still denote it by x) $x \in \Delta_k$. Accordingly, the MVL function f in (1) becomes $f : \prod_{i=1}^n \Delta_{k_i} \to \Delta_{k_0}$, which is called the vector form of f.

The following result is fundamental, which assigns to each MVL function a unique logical matrix.

Theorem 4 (*Cheng et al., 2011*). *Given an MVL function $f : \prod_{i=1}^n \Delta_{k_i} \to \Delta_{k_0}$. Then there exists a unique matrix $M_f \in \mathcal{L}_{k_0\times k}$, called the structure matrix of f, such that f can be expressed into its algebraic form as*

$$f(x_1, \dots, x_n) = M_f x, \tag{7}$$

where $x = \ltimes_{i=1}^n x_i$, $k = \prod_{i=1}^n k_i$.

We give an example to describe how to construct M_f.

Example 5. Assume that an MVL function $f : \mathcal{D}_3 \times \mathcal{D}_2 \to \mathcal{D}_3$ is defined by Table 1, where $z = f(x, y)$. Then in vector forms $x, z \in \Delta_3$, and $y \in \Delta_2$. According to Table 1, we have $f(\delta_3^1, \delta_2^1) = \delta_3^2$, $f(\delta_3^1, \delta_2^2) = \delta_3^1$, and so on. Noting (7), the first column of M_f is δ_3^2, the second column is δ_3^1 and so on. Finally, we have $M_f = \delta_3[2, 1, 2, 3, 1, 2]$.

The structure matrices of some useful gates are described in the following example.

Example 6. By Theorem 4, a 2-gate F corresponds to a logical matrix $M_F \in \mathcal{L}_{2\times 4}$. Hence, there are $2^4 = 16$ possible M_F, for instance,

(i) max-gate: $F(x, y) := x \vee y$, $M_\vee = \delta_2[1\ 1\ 1\ 2]$;
(ii) min-gate: $F(x, y) := x \wedge y$, $M_\wedge = \delta_2[1\ 2\ 2\ 2]$;
(iii) imply-gate: $F(x, y) := x \to y$, $M_\to = \delta_2[1\ 2\ 1\ 1]$.

Similarly, a 3-gate F corresponds to a logical matrix $M_F \in \mathcal{L}_{3\times 9}$. Hence, there are $3^9 = 19\,683$ possible M_F. For instance, if we denote by $\vee$ and $\wedge$ the max-gate and min-gate, respectively, then $M_\vee = \delta_3[1\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 3]$ with the max-gate $F(x, y) = \max\{x, y\}$, and $M_\wedge = \delta_3[1\ 2\ 3\ 2\ 2\ 3\ 3\ 3\ 3]$ with the min-gate $F(x, y) = \min\{x, y\}$.

Remark 7. The calculations involved in converting a logical mapping into its algebraic form and back from an algebraic form to a set of logical functions etc. are standard. An STP toolbox is provided in http://lsc.amss.ac.cn/dcheng for the related computations, and we refer the reader to Cheng et al. (2011) for details.

3. Disjoint bi-decomposition

Definition 8. Let $f : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ be an MVL function as in (1). Let $\{\Gamma, \Lambda\}$ be a partition of $\{1, 2, \ldots, n\}$. f is said to be bi-decomposable with respect to Γ and Λ if there exist an operator function $F : \mathcal{D}_{k_0}^2 \to \mathcal{D}_{k_0}$, two decomposition functions $\phi : \prod_{i\in\Gamma} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ and $\psi : \prod_{i\in\Lambda} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$, such that

$$f(x_1, \ldots, x_n) = F(\phi(x_\gamma), \psi(x_\lambda)), \tag{8}$$

with $x_\gamma \in \prod_{i\in\Gamma} \mathcal{D}_{k_i}$, and $x_\lambda \in \prod_{i\in\Lambda} \mathcal{D}_{k_i}$.

Throughout this paper we assume the partition is well ordered. That is,

$$\Gamma = \{1, 2, \ldots, r\}, \qquad \Lambda = \{r + 1, r + 2, \ldots, n\}. \tag{9}$$

In general, a variable re-ordering gives this well-ordered partition.

Next, we introduce a new concept called *type*.

Definition 9. Given an integer $t \geq 2$, a t-type is a set of t logical matrices of dimension $t \times t$. That is,

$$T = \left\{T_1, T_2, \ldots, T_t \middle| T_i \in \mathcal{L}_{t\times t},\ 1 \leq i \leq t\right\}.$$

Remark 10. Let F be a t-gate with its structure matrix as $M_F = [M_1\ M_2\ \cdots\ M_t]$, where $M_i \in \mathcal{L}_{t\times t}$, $i = 1, \ldots, t$. Then a t-type generated by F is $T_F = \{M_1, M_2, \ldots, M_t\}$. Conversely, a t-type corresponds to a set of t-gates, which can be constructed by putting matrices in the t-type into a row. It will be shown later that the order of M_i in M_F does not affect the decomposability, and thus M_F can be constructed simply by putting M_i in the same order as it appears in T_F.

Example 11. Consider Example 6 again. The 2-types corresponding to $M_\vee$ and $M_\wedge$ are $T_\vee = \{\delta_2[1\ 1], \delta_2[1\ 2]\}$ and $T_\wedge = \{\delta_2[1\ 2], \delta_2[2\ 2]\}$, respectively.

Similarly, the 3-types corresponding to $M_\vee$ and $M_\wedge$ are $T_\vee = \{\delta_3[1\ 1\ 1], \delta_3[1\ 2\ 2], \delta_3[1\ 2\ 3]\}$, and $T_\wedge = \{\delta_3[1\ 2\ 3], \delta_3[2\ 2\ 3], \delta_3[3\ 3\ 3]\}$, respectively.

Now consider partition (9). Denote $p = \prod_{i=1}^r k_i$, $q = \prod_{i=r+1}^n k_i$, $k = pq$, and the structure matrix of MVL function f in (1) as

$$M_f = [M_1\ M_2\ \cdots\ M_p], \tag{10}$$

where $M_i \in \mathcal{L}_{k_0\times q}$, $i = 1, \ldots, p$.

Then we have the following theorem.

Theorem 12. *Consider the MVL function (1) with its structure matrix as (10). It is bi-decomposable with respect to the partition (9), if and only if there exist* (i) *a k_0-type $T = \{T_1, T_2, \ldots, T_{k_0}\} \subset \mathcal{L}_{k_0\times k_0}$, and* (ii) *a logical matrix $M_\psi \in \mathcal{L}_{k_0\times q}$, such that*

$$M_i = T_{s_i} M_\psi, \quad \text{where } T_{s_i} \in T, i = 1, \ldots, p. \tag{11}$$

Table 2
Possible 2-types with their corresponding functions.

$\{\mathfrak{T}_1, \mathfrak{T}_1\}$	$\{\mathfrak{T}_1, \mathfrak{T}_2\}$	$\{\mathfrak{T}_1, \mathfrak{T}_3\}$	$\{\mathfrak{T}_1, \mathfrak{T}_4\}$	$\{\mathfrak{T}_2, \mathfrak{T}_2\}$
1	$\phi \vee \psi$	$\phi \vee \neg\psi$	ϕ	ψ
$\{\mathfrak{T}_2, \mathfrak{T}_3\}$	$\{\mathfrak{T}_2, \mathfrak{T}_4\}$	$\{\mathfrak{T}_3, \mathfrak{T}_3\}$	$\{\mathfrak{T}_3, \mathfrak{T}_4\}$	$\{\mathfrak{T}_4, \mathfrak{T}_4\}$
$\phi \leftrightarrow \psi$	$\phi \wedge \psi$	$\neg\psi$	$\neg(\phi \to \psi)$	0

Proof. (Necessity). Assume that there are three functions F, ϕ, and ψ, such that (8) holds. Denote the structure matrix of f by $M_f \in \mathcal{L}_{k_0\times k}$, which is split as in (10). Denote the structure matrix of F, ϕ and ψ by $M_F = \left[F_1\ F_2\ \cdots\ F_{k_0}\right]$, $M_\phi = \delta_{k_0}\left[i_1\ i_2\ \cdots\ i_p\right]$ and $M_\psi \in \mathcal{L}_{k_0\times q}$, respectively, where $F_i \in \mathcal{L}_{k_0\times k_0}$, $i = 1, \ldots, k_0$.

Let $x := \ltimes_{i=1}^n x_i$, $x^1 := \ltimes_{i=1}^r x_i$, and $x^2 := \ltimes_{i=r+1}^n x_i$, then $M_f x = M_F M_\phi x^1 M_\psi x^2$. Using Proposition 3, we have that

$$M_f = M_F M_\phi \left(I_p \otimes M_\psi\right). \tag{12}$$

We first calculate $M_F M_\phi$, which is denoted by $M_F M_\phi := [N_1\ N_2\ \cdots\ N_p]$. Then a straightforward computation shows that $N_j = F_{i_j}$, $j = 1, \ldots, p$. If we denote

$$M_F M_\phi \left(I_p \otimes M_\psi\right) := \left[W_1\ W_2\ \cdots\ W_p\right], \tag{13}$$

then

$$W_j = F_{i_j} M_\psi, \quad j = 1, \ldots, p. \tag{14}$$

Substituting the left and right hand sides of (12) by (10) and (13)–(14) respectively, and comparing corresponding blocks yield

$$M_j = F_{i_j} M_\psi, \quad j = 1, \ldots, p. \tag{15}$$

Setting $T = \left\{F_1, F_2, \ldots, F_{k_0}\right\}$, (15) proves the necessity.

(Sufficiency). Assume that there exist a k_0-type T and a matrix M_ψ such that (11) holds. Define k_0-gate F by its structure matrix as $M_F = \left[T_1\ T_2\ \cdots\ T_{k_0}\right]$, and $M_\phi = \delta_{k_0}\left[s_1\ s_2\ \cdots\ s_p\right]$. Then a straightforward computation shows that (12) holds, which implies that $f(x) = F(\phi(x^1), \psi(x^2))$. □

Applying Theorem 12 to the Boolean case, we have the following corollary.

Corollary 13. *Consider a Boolean function $f(x_1, \ldots, x_n)$. It is bi-decomposable with respect to the partition (9), if and only if there exist a 2-type T and a logical matrix $M_\psi \in \mathcal{L}_{2\times 2^{n-r}}$ such that the structure matrix of f has the form as*

$$M_f = \left[\mu_1 M_\psi\ \mu_2 M_\psi\ \cdots\ \mu_{2^r} M_\psi\right],$$

where $\mu_i \in T$, $\forall i$.

Remark 14. Note that $\mathcal{L}_{2\times 2} = \left\{\mathfrak{T}_1 := \delta_2[1, 1], \mathfrak{T}_2 := \delta_2[1, 2], \mathfrak{T}_3 := \delta_2[2, 1], \mathfrak{T}_4 := \delta_2[2, 2]\right\}$. Ignoring the order of two components, we have 10 possible 2-types $T = \{\mathfrak{T}_i, \mathfrak{T}_j\}$, $1 \leq i, j \leq 4$, with their logical forms given in Table 2.

Remark 15. Theorem 2.1 in Sasao and Butler (1989), a result for the disjoint bi-decomposition of Boolean functions, says that "f has a disjoint bi-decomposition of form $f(X_1, X_2) = h(g_1(X_1), g_2(X_2))$, if and only if $\mu(f : X_1, X_2) \leq 2$ and $\mu(f : X_2, X_1) \leq 2$". Actually, for a 2-type $T = \{T_1, T_2\}$, $\mu(f : X_1, X_2) = 1(2)$ is equivalent to $T_1 = T_2 (T_1 \neq T_2)$, and $\mu(f : X_2, X_1) = 1(2)$ is equivalent to $T_1, T_2 \in \{\mathfrak{T}_1, \mathfrak{T}_4\} (\exists i \in \{1, 2\}, T_i \in \{\mathfrak{T}_2, \mathfrak{T}_3\})$, using notations of Remark 14. Specifically, we have the following correspondences.

(i) $\mu(f : X_1, X_2) = 1$ and $\mu(f : X_2, X_1) = 1$ corresponds to $\{\mathfrak{T}_1, \mathfrak{T}_1\}, \{\mathfrak{T}_4, \mathfrak{T}_4\}$;
(ii) $\mu(f : X_1, X_2) = 1$ and $\mu(f : X_2, X_1) = 2$ corresponds to $\{\mathfrak{T}_2, \mathfrak{T}_2\}, \{\mathfrak{T}_3, \mathfrak{T}_3\}$;

(iii) $\mu(f : X_1, X_2) = 2$ and $\mu(f : X_2, X_1) = 1$ corresponds to $\{\mathfrak{T}_1, \mathfrak{T}_4\}$;
(iv) $\mu(f : X_1, X_2) = 2$ and $\mu(f : X_2, X_1) = 2$ corresponds to the rest types.

Therefore, this theorem and Corollary 13 are essentially the same. Furthermore, Corollary 13 is also practically the same as Theorem 11.5 in Davio et al. (1978).

Similarly, one can examine that Theorem 1 in Waliuzzaman and Vranesic (1970), which deals with the Ashenhurst-decomposition of k-valued functions, can also be seen as a special case of Theorem 12.

Remark 16. Theoretically, since Theorem 12 involves only finite sets, it is verifiable. But practically, exhaustive searching is a very heavy job. We propose the following procedure to verify the conditions.

(i) Step 1. Split M_f into p equal blocks as in (10).
(ii) Step 2. Define matrix
$$\Psi := \begin{bmatrix} M_1^T & M_2^T & \cdots & M_p^T \end{bmatrix}^T. \tag{16}$$
Then check whether $\mathrm{rank}(\Psi) \le k_0$. If "No", then the function is not decomposable (with respect to this p and the partition). Stop. (Note that $\mathrm{rank}(\Psi) \le k_0$ comes from (11).) If "Yes", go to the next step.
(iii) Step 3. If there exist independent rows of Ψ to construct a logical matrix of dimension $k_0 \times q$, let the matrix be M_ψ and go to the next step.
(iv) Step 4. With such M_ψ, solve the algebraic equations (11) for T_{s_i}, $i = 1, \ldots, p$. If the solution exists with $T_{s_i} \in \mathcal{L}_{k_0\times k_0}$, and there are totally $s \le k_0$ different solutions (i.e., $\left|\{T_{s_i} | i = 1, \ldots, p\}\right| \le k_0$), then the decomposition exists.

The following example illustrates Theorem 12 and Remark 16.

Example 17. For simplicity, we consider the case of a 3-valued function. Let $f(x_1, x_2, x_3, x_4) : \mathcal{D}_3^4 \to \mathcal{D}_3$ be a logical function with its structure matrix M_f equals

$$\begin{aligned}\delta_3[&1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\quad 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\quad 1\ 2\ 3\ 2\ 2\ 2\ 3\ 2\ 1\\ &1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\quad 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\quad 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1\\ &1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\quad 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\quad 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1].\end{aligned}$$

Consider the decomposition with partition $\Gamma = \{1, 2\}$, $\Lambda = \{3, 4\}$, then $p = q = 9$, $M_f = [M_1\ M_2\ \cdots\ M_9]$ and Ψ is given by (16).

It is easy to check $\mathrm{rank}(\Psi) = k_0 = 3$, and we can choose linearly independent rows of Ψ to form logical matrix M_ψ. For example, take the 7, 8, 9-th rows of Ψ (i.e., M_3), then $M_\psi = \delta_3[1\ 2\ 3\ 2\ 2\ 2\ 3\ 2\ 1]$.

Next, we solve (11) for T_1, T_2, T_3, that is,

$$\begin{aligned}T_1 M_\psi &= \delta_3[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1],\\ T_2 M_\psi &= \delta_3[1\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 1],\\ T_3 M_\psi &= \delta_3[1\ 2\ 3\ 2\ 2\ 2\ 3\ 2\ 1].\end{aligned}$$

One valid solution is $T = \{T_1 = \delta_3[1\ 1\ 1],\ T_2 = \delta_3[1\ 2\ 2],\ T_3 = \delta_3[1\ 2\ 3]\}$, and therefore we can choose

$$M_F = \delta_3[1\ 1\ 1\quad 1\ 2\ 2\quad 1\ 2\ 3].$$

Finally, since $M_f = [T_1, T_2, T_3, T_1, T_2, T_2, T_1, T_1, T_1](I_9 \otimes M_\psi)$, we have $M_\phi = \delta_3[1\ 2\ 3\quad 1\ 2\ 2\quad 1\ 1\ 1]$. If we define the 3-valued logical operators $\to$ and $\leftrightarrow$ by using the basic operators $\{\vee, \wedge, \neg\}$ as follows:

$$\begin{aligned}A \to B &:= (A \wedge B) \vee \neg A,\\ A \leftrightarrow B &:= (A \to B) \wedge (B \to A),\end{aligned}$$

then it is easy to verify that $M_\to = M_\phi$, and $M_\leftrightarrow = M_\psi$. Eventually, we have function f in the decomposed form as $f(x_1, x_2, x_3, x_4) = (x_1 \to x_2) \vee (x_3 \leftrightarrow x_4)$.

4. Non-disjoint bi-decomposition

Definition 18. Let $f : \prod_{i=1}^n \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ be an MVL function as in (1), and $\{\Gamma, \Theta, \Lambda\}$ be a partition of $\{1, 2, \ldots, n\}$. f is said to be bi-decomposable with respect to $\Gamma \cup \Theta$ and $\Lambda \cup \Theta$ if there exist an operator function $F : \mathcal{D}_{k_0}^2 \to \mathcal{D}_{k_0}$, two decomposition functions $\phi : \prod_{i\in\Gamma\cup\Theta} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$, and $\psi : \prod_{i\in\Lambda\cup\Theta} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$, such that

$$f(x_1, \ldots, x_n) = F\left(\phi(x_\gamma), \psi(x_\lambda)\right), \tag{17}$$

where $x_\gamma \in \prod_{i\in\Gamma\cup\Theta} \mathcal{D}_{k_i}$, and $x_\lambda \in \prod_{i\in\Lambda\cup\Theta} \mathcal{D}_{k_i}$.

Similar to the disjoint case, we assume the partition is well-ordered. That is, there exist $r_1 \ge 1$, $r_2 \ge 1$ with $r_1 + r_2 < n$, such that

$$\begin{aligned}\Gamma &= \{1, 2, \ldots, r_1\},\\ \Theta &= \{r_1+1, r_1+2, \ldots, r_1+r_2\},\\ \Lambda &= \{r_1+r_2+1, r_1+r_2+2, \ldots, n\}.\end{aligned} \tag{18}$$

For statement ease, denote $\prod_{i=1}^{r_1} k_i = p$, $\prod_{i=r_1+1}^{r_1+r_2} k_i = \ell$, $\prod_{i=r_1+r_2+1}^{n} k_i = q$, and denote the structure matrix of f by M_f, which can be split into ℓp equal blocks as

$$M_f = [M_1^1\ M_1^2\ \cdots\ M_1^\ell\ \ \cdots\ M_p^1\ M_p^2\ \cdots\ M_p^\ell]. \tag{19}$$

Now we are ready to present the following theorem.

Theorem 19. *Consider an MVL function (1) with its structure matrix M_f. f can be decomposed as in (17) with respect to the partition (18), if and only if there exist* (i) *a k_0-type T, and* (ii) *a set of logical matrices $M_\psi^s \in \mathcal{L}_{k_0\times q}$, $s = 1, \ldots, \ell$, such that the structure matrix of f can be expressed as*

$$M_f = \begin{bmatrix}\mu_{1,1}M_\psi^1 & \mu_{1,2}M_\psi^2 & \cdots & \mu_{1,\ell}M_\psi^\ell\\ \mu_{2,1}M_\psi^1 & \mu_{2,2}M_\psi^2 & \cdots & \mu_{2,\ell}M_\psi^\ell\\ \vdots & & \ddots & \vdots\\ \mu_{p,1}M_\psi^1 & \mu_{p,2}M_\psi^2 & \cdots & \mu_{p,\ell}M_\psi^\ell\end{bmatrix} \tag{20}$$

where $\mu_{i,j} \in T$, $i = 1, \ldots, p$, $j = 1, \ldots, \ell$.

Proof. (Necessity). Since there are three functions F, ϕ, and ψ such that (17) holds, we denote the structure matrix of F by $M_F = \left[F_1\ F_2\ \cdots\ F_{k_0}\right]$, where $F_i \in \mathcal{L}_{k_0\times k_0}$, $i = 1, \ldots, k_0$; the structure matrix of ϕ by $M_\phi = \delta_{k_0}[j_1\ j_2\ \cdots\ j_{p\ell}]$; and the structure matrix of ψ by $M_\psi = \left[M_\psi^1\ M_\psi^2\ \cdots\ M_\psi^\ell\right] \in \mathcal{L}_{k_0\times q\ell}$, where $M_\psi^i \in \mathcal{L}_{k_0\times q}$, $i = 1, \ldots, \ell$. Then construct a k_0-type T as $T := \left\{F_1, F_2, \ldots, F_{k_0}\right\}$.

According to (17), we have

$$M_f x = M_F M_\phi x^1 x^2 M_\psi x^2 x^3,$$

where $x = \ltimes_{i=1}^n x_i$, $x^1 = \ltimes_{i=1}^{r_1} x_i$, $x^2 = \ltimes_{i=r_1+1}^{r_1+r_2} x_i$, $x^3 = \ltimes_{i=r_1+r_2+1}^{n} x_i$.

Using formulas (5) and (6) in Proposition 3, we have that

$$M_f = M_F M_\phi \left(I_{p\ell} \otimes M_\psi\right)\left(I_p \otimes R_\ell\right), \tag{21}$$

where R_ℓ is defined in (6).

We first calculate $M_F M_\phi$, which is denoted as

$$M_F M_\phi := [N_1\ N_2\ \cdots\ N_{p\ell}]. \tag{22}$$

Similar to the disjoint case, we have $N_s \in T$, $s = 1, 2, \ldots, p\ell$. Next, we calculate $\left(I_{p\ell} \otimes M_\psi\right)\left(I_p \otimes R_\ell\right)$:

$$\left(I_{p\ell} \otimes M_\psi\right)\left(I_p \otimes R_\ell\right) = I_p \otimes \left[\left(I_\ell \otimes M_\psi\right) R_\ell\right]. \tag{23}$$

We simplify $\left(I_\ell \otimes M_\psi\right) R_\ell$ first. Note that $\left(I_\ell \otimes M_\psi\right) \in \mathcal{L}_{\ell k_0\times \ell^2 q}$ and $R_\ell \in \mathcal{L}_{\ell^2\times\ell}$. Converting them back to the conventional matrix product we have

$$\left(I_\ell \otimes M_\psi\right) R_\ell = \left(I_\ell \otimes M_\psi\right)\left(R_\ell \otimes I_q\right). \tag{24}$$

Moreover, we have

$$I_\ell \otimes M_\psi = diag(\underbrace{M_\psi, \ldots, M_\psi}_{\ell}); \tag{25}$$

and

$$R_\ell \otimes I_q = diag\left(\left.\begin{bmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}\ell, \left.\begin{bmatrix} 0 \\ I_q \\ \vdots \\ 0 \end{bmatrix}\right\}\ell, \ldots, \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_q \end{bmatrix}\right\}\ell\right). \tag{26}$$

Multiplying (25) with (26) yields

$$\left(I_\ell \otimes M_\psi\right) R_\ell = diag(M_\psi^1, M_\psi^2, \ldots, M_\psi^\ell). \tag{27}$$

Putting (22)–(24) and (27) together, the right-hand-side of (21) can be calculated, and it is ready to verify that the right-hand-side of (21) has the form of (20). The necessity part is thus proved.

(Sufficiency). For the k_0-type $T = \{T_1, T_2, \ldots, T_{k_0}\}$, we construct its corresponding k_0-gate F by $M_F = [T_1\ T_2\ \cdots\ T_{k_0}]$. Use $M_\psi = [M_\psi^1\ M_\psi^2\ \cdots\ M_\psi^\ell]$ as the structure matrix of ψ. Denote

$$M_\phi = \left[M_\phi^{1,1}\ \cdots\ M_\phi^{1,\ell}\ \ \cdots\ \ M_\phi^{p,1}\ \cdots\ M_\phi^{p,\ell}\right].$$

Since (20) is known, according to $\mu_{\alpha,\beta}$, we can determine $M_\phi^{\alpha,\beta}$ as follows: if $\mu_{\alpha,\beta} = T_i$, then

$$M_\phi^{\alpha,\beta} = \delta_{k_0}^i. \tag{28}$$

Since the structure matrix of a logical function uniquely determines the function, we can have the pair of decomposition functions ϕ, ψ via the structure matrices M_ϕ, M_ψ obtained above. Then, it is easy to check that the factorization (17) holds (via (21)). □

Remark 20. By the proof of Theorems 12 and 19, it can be readily seen that as long as type T exists, the order of T_i to form M_F does not affect the decomposability but only induces different M_ϕ.

Remark 21. Applying Theorem 19 to the Boolean function case, again we can find that our result coincides with Theorem 3.2 in Sasao and Butler (1989). Therefore, our explicit decomposition expression for the MVL function in Theorem 19 is a generalization of the implicit form for the Boolean function in Sasao and Butler (1989).

Remark 22. Similar to Remark 16, we give a procedure to verify Theorem 19.

(i) Step 1. Split M_f into ℓp equal blocks as given in (19).
(ii) Step 2. Define ℓ matrices

$$\Psi_j := \left[(M_1^j)^T\quad (M_2^j)^T\quad \cdots\quad (M_p^j)^T\right]^T, \quad j = 1, \ldots, \ell.$$

Then check whether $\text{rank}(\Psi_j) \le k_0$, $\forall j$. If "No", then the function is not decomposable (with respect to this p and the partition). Stop. If "Yes", go to the next step.
(iii) Step 3. Use the independent rows of Ψ_j to construct logical matrices $M_\psi^j \in \mathcal{L}_{k_0\times q}$, $j = 1, \ldots, \ell$, and go to the next step.
(iv) Step 4. With those M_ψ^j, $j = 1, \ldots, \ell$, solve the algebraic equations $\mu_{i,j} M_\psi^j = M_i^j$, $i = 1, \ldots, p$; $j = 1, \ldots, \ell$. If the solution exists with $\mu_{i,j} \in \mathcal{L}_{k_0\times k_0}$, and there are totally $s \le k_0$ different solutions (i.e., $\left|\{\mu_{i,j} | i = 1, \ldots, p;\ j = 1, \ldots, \ell\}\right| \le k_0$), then the considered decomposition exists.

We use the following simple example to illustrate this.

Example 23. Let $f(x_1, x_2, x_3, x_4) : \mathcal{D}_3^4 \to \mathcal{D}_3$ be a 3-valued logical function with its structure matrix M_f as

$$\begin{aligned}\delta_3[&1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ \ 2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ \ 3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3\\ &1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\\ &1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ \ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ \ 1\ 2\ 3\ 1\ 2\ 2\ 1\ 1\ 1].\end{aligned}$$

We consider the decomposition with partition: $\Gamma = \{1\}$, $\Theta = \{2\}$, and $\Lambda = \{3, 4\}$. Then $p = \ell = 3$, $q = 9$. Split M_f as $M_f = [M_1^1\ M_1^2\ M_1^3\ \ M_2^1\ M_2^2\ M_2^3\ \ M_3^1\ M_3^2\ M_3^3]$. Then we have

$$\Psi_1 = \begin{bmatrix} \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3] \\ \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3] \\ \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3] \end{bmatrix};$$

$$\Psi_2 = \begin{bmatrix} \delta_3[2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3] \\ \delta_3[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \\ \delta_3[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \end{bmatrix};$$

$$\Psi_3 = \begin{bmatrix} \delta_3[3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3] \\ \delta_3[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \\ \delta_3[1\ 2\ 3\ 1\ 2\ 2\ 1\ 1\ 1] \end{bmatrix}.$$

It is ready to check that $\text{rank}(\Psi_i) = k_0 = 3, i = 1, 2, 3$. Then, we can choose linearly independent rows of Ψ_i to form M_ψ^i for $i = 1, 2, 3$. For example, take $M_\psi^1 = \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3]$, $M_\psi^2 = \delta_3[2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3]$, $M_\psi^3 = \delta_3[3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3]$, and then $M_\psi = [M_\psi^1, M_\psi^2, M_\psi^3] = \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ 2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3\ \ 3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3]$. Next, we solve the following equations:

$$\begin{aligned}
\mu_{i,1}M_\psi^1 &= \delta_3[1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 3], \quad i = 1, 2, 3;\\
\mu_{1,2}M_\psi^2 &= \delta_3[2\ 2\ 1\ 2\ 2\ 2\ 3\ 3\ 3],\\
\mu_{i,2}M_\psi^2 &= \delta_3[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2], \quad i = 2, 3;\\
\mu_{1,3}M_\psi^3 &= \delta_3[3\ 2\ 1\ 3\ 2\ 2\ 3\ 3\ 3],\\
\mu_{2,3}M_\psi^3 &= \delta_3[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2],\\
\mu_{3,3}M_\psi^3 &= \delta_3[1\ 2\ 3\ 1\ 2\ 2\ 1\ 1\ 1].
\end{aligned}$$

One valid solution is $\mu_{1,1} = \mu_{2,1} = \mu_{3,1} = \mu_{1,2} = \mu_{1,3} = \delta_3[1\ 2\ 3]$; $\mu_{2,2} = \mu_{2,3} = \mu_{3,2} = \delta_3[2\ 2\ 2]$; $\mu_{3,3} = \delta_3[3\ 2\ 1]$. Hence, we can choose $T = \{T_1 = \delta_3[1\ 2\ 3],\ T_2 = \delta_3[2\ 2\ 2],\ T_3 = \delta_3[3\ 2\ 1]\}$ and construct $M_F = \delta_3[1\ 2\ 3\ \ 2\ 2\ 2\ \ 3\ 2\ 1]$. Finally, according to (28), we have $M_\phi = \delta_3[1\ 1\ 1\ 1\ 2\ 2\ 1\ 2\ 3]$. Thus, $f(x_1, x_2, x_3, x_4) = M_f x = M_F M_\phi x_1 x_2 M_\psi x_2 x_3 x_4$, with M_F, M_ϕ, M_ψ obtained as above. Converting back to the logical form, we have

$$f(x_1, x_2, x_3, x_4) = (x_1 \vee x_2) \leftrightarrow [(x_2 \vee \neg x_4) \wedge x_3].$$

5. Implicit function theorem

Consider a set of $r(< n)$ k-valued logical equations:

$$g_j(x_1, \ldots, x_n) = 1, \quad j = 1, \ldots, r, \tag{29}$$

where $g_j : \mathcal{D}_k^n \to \mathcal{D}_k$ is a k-valued logical function, and $x_i \in \mathcal{D}_k$, $i = 1, \ldots, n$. Note that the right hand side of (29) can be any constant $c_j \in \mathcal{D}_k$, but we choose $c_j = 1 \sim \delta_k^1$ for convenience. In the vector form, we set $x^1 = \ltimes_{i=1}^{n-r} x_i \in \Delta_p$, where $p = k^{n-r}$, $x^2 = \ltimes_{i=n-r+1}^{n} x_i \in \Delta_q$, where $q = k^r$. Let $g = (g_1, \ldots, g_r)$, then g is a logical mapping $\mathcal{D}_k^n \to \mathcal{D}_k^r$, or $\Delta_p \times \Delta_q \to \Delta_q$ in the vector form. The set of r equations of (29) can now be expressed into its algebraic form as

$$M_g x^1 x^2 = \delta_q^1, \tag{30}$$

where M_g is the structure matrix of g.

This section considers when x^2 can be solved as functions of x^1, or more precisely, when (29) can be expressed as

$$x_j = \phi_j(x_1, \ldots, x_{n-r}), \quad j = n - r + 1, \ldots, n, \tag{31}$$

where $\phi_j : \mathcal{D}_k^{n-r} \to \mathcal{D}_k$ is an appropriate k-valued function. Let $\phi = (\phi_{n-r+1}, \ldots, \phi_n)$, then ϕ is a logical mapping $\mathcal{D}_k^{n-r} \to \mathcal{D}_k^r$, or equivalently, $\Delta_p \to \Delta_q$.

The answer is the Implicit Function Theorem (IFT) for k-valued functions. In fact, the problem can be considered as a special bi-decomposition of MVL functions. To see that, for $x, y \in \mathcal{D}_q$, we define a q-gate "$\leftrightarrow$" as

$$x \leftrightarrow y = \begin{cases} 1, & \text{if } x = y, \\ \xi, & \text{if } x \neq y, \end{cases} \tag{32}$$

where ξ is a constant in $\mathcal{D}_q$ and $\xi \neq 1$. Then (31) can be expressed as

$$F_{\leftrightarrow}(\phi(x^1), x^2) = 1. \tag{33}$$

It is clear that to make (29) and (31) equivalent, it is necessary and sufficient that

$$g(x^1, x^2) = F_{\leftrightarrow}(\phi(x^1), x^2). \tag{34}$$

This is the basic idea for deducing global IFT from the viewpoint of the bi-decomposition of MVL functions.

Note that ξ in (32) can be an arbitrary constant in $\mathcal{D}_q$, which means we may have a set of operators $\leftrightarrow$ that satisfies (34). Hence, we have to construct the set of all types that realize $\leftrightarrow$.

For a positive integer $q > 1$, define a set of matrices as

$$\Xi_i = \left\{ E_i \in \mathcal{L}_{q\times q} \middle| \mathrm{Col}_i(E_i) = \delta_q^1;\ \mathrm{Col}_j(E_i) \neq \delta_q^1, j \neq i \right\}$$

where $i = 1, 2, \ldots, q$. Using Ξ_i, we construct a set of q-types as

$$\mathcal{E}_q := \left\{ \{E_1, \ldots, E_q\} \middle| E_i \in \Xi_i, i = 1, \ldots, q \right\}. \tag{35}$$

As in previous sections, each type $T \in \mathcal{E}_q$ corresponds to a q-gate $F : \mathcal{D}_q \times \mathcal{D}_q \to \mathcal{D}_q$, which has $E_T := \left[E_1\ E_2\ \cdots\ E_q\right]$ as its structure matrix.

The following lemma shows that $\mathcal{E}_q$ is the set of types corresponding to $\leftrightarrow$.

Lemma 24. *Let $x, y \in \Delta_q$. Then $x = y$ if and only if there exists a $T \in \mathcal{E}_q$ such that*

$$E_T xy = \delta_q^1. \tag{36}$$

Proof. Assume that $x = \delta_q^\alpha$ and $y = \delta_q^\beta$. Then a straightforward computation shows that $E_T xy = \mathrm{Col}_\beta(E_\alpha)$, where $E_T = [E_1\ E_2\ \cdots\ E_q]$. Hence (36) holds if and only if $\alpha = \beta$. □

Then we have the following theorem.

Theorem 25 (*Implicit Function Theorem*). *Suppose the structure matrix of g associated with (29) can be expressed as $M_g = [M_1\ M_2, \ldots, M_q]$. Then x_j ($j = n - r + 1, \ldots, n$) can be solved as (31) from (29), if and only if there exists a q-type $T = \{E_1, E_2, \ldots, E_q\} \in \mathcal{E}_q$, such that*

$$M_i \in T, \quad i = 1, \ldots, q. \tag{37}$$

Proof. Express (31) in its algebraic form as $x^2 = M_\phi x^1$. According to Lemma 24, (29) can be expressed as (31), if and only if there exists a $T \in \mathcal{E}_q$ such that its corresponding q-gate F with $M_F = E_T$ satisfies

$$M_F M_\phi x^1 x^2 = \delta_q^1. \tag{38}$$

Comparing Eq. (38) with (30), it is clear that the necessary and sufficient condition becomes that g can be expressed as

$$M_g x^1 x^2 = M_F M_\phi x^1 x^2, \tag{39}$$

where M_F corresponds to $T \in \mathcal{E}_q$. Now formally consider $x^2 = M_\psi x^2$ with $M_\psi = I_q$, then the conclusion follows from Lemma 24 and Theorem 12 immediately. □

Remark 26. The Implicit Function Theorem (IFT) for Boolean functions was discussed in Bazso (2000), where a sufficient condition for local IFT was given. Theorem 25 is a global one and for general k-valued functions.

6. Normalization of D–A Boolean networks

As an application of IFT, we consider the dynamic–algebraic Boolean networks. Consider a Boolean network of n nodes. Assume that there are $n - r$ nodes, which satisfy Boolean dynamic models as

$$x_i(t+1) = f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n - r, \tag{40}$$

where $x_i \in \mathcal{D}$, $i = 1, \ldots, n$. Besides, the other r nodes are determined by certain algebraic (logical) equations as (29), which may be the constraints posed on these nodes. We call (40)–(29) a dynamic–algebraic (D–A) Boolean network.

Obviously, if we can obtain (31) from the set of equations (29), the D–A Boolean network can be transformed into a standard Boolean network by substituting (31) into (40). We call this process the *normalization* of D–A Boolean networks. Clearly, the IFT obtained in the last section is crucial to the normalization.

We give an example to depict this.

Example 27. Consider the following dynamic–algebraic Boolean network:

$$\begin{cases} x_1(t+1) = x_2(t) \to x_4(t), \\ x_2(t+1) = x_1(t) \wedge x_3(t), \\ 1 = (x_3(t) \bar{\vee} x_4(t)) \leftrightarrow (x_1(t) \bar{\vee} x_2(t)), \\ 0 = x_4(t) \bar{\vee} (x_1(t) \vee x_2(t)). \end{cases} \tag{41}$$

We intend to solve x_3 and x_4 out from the last two equations. First, we convert them to

$$\begin{cases} g_1(x_1, x_2, x_3, x_4) := (x_3(t) \bar{\vee} x_4(t)) \leftrightarrow (x_1(t) \bar{\vee} x_2(t)) = 1 \\ g_2(x_1, x_2, x_3, x_4) := x_4(t) \leftrightarrow (x_1(t) \vee x_2(t)) = 1, \end{cases}$$

where g_2 is the negation ($\neg$) of the last equation in (41).

Converting them into the vector form:

$$\begin{cases} M_{g_1} x = \delta_2[1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1]x \\ M_{g_2} x = \delta_2[1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 1]x. \end{cases} \tag{42}$$

Then the structure matrix of $g = (g_1, g_2)$ can be easily calculated as $M_g = \delta_4[1\ 4\ 3\ 2\ \ 3\ 2\ 1\ 4\ \ 3\ 2\ 1\ 4\ \ 2\ 3\ 4\ 1]$. Recalling Section 3, a 4-type $T = \{E_1, E_2, E_3, E_4\} \in \mathcal{E}_4$ satisfying (37) can be obtained, where $E_1 = \delta_4[1\ 4\ 3\ 2]$, $E_3 = \delta_4[3\ 2\ 1\ 4]$, $E_4 = [2\ 3\ 4\ 1]$ and E_2 can be arbitrary. Then, we have $M_\phi = \delta_4[1\ 3\ 3\ 4]$, which means $x_3(t)x_4(t) = \delta_4[1\ 3\ 3\ 4]x_1(t)x_2(t)$. It follows that $x_3(t)$ and $x_4(t)$ can be solved from (42) uniquely as

$$\begin{cases} x_3(t) = x_1(t) \wedge x_2(t) \\ x_4(t) = x_1(t) \vee x_2(t). \end{cases} \tag{43}$$

Substituting (43) into (41) yields the normal form of (41):

$$\begin{cases} x_1(t+1) = x_2(t) \to (x_1(t) \vee x_2(t)) \\ x_2(t+1) = x_1(t) \wedge x_2(t). \end{cases} \tag{44}$$

Then, with algebraic equations (43) for x_3, x_4, the dynamics of the D–A Boolean network (41) is determined by (44).

Remark 28. The method provided above can also be used for the control problems of dynamic–algebraic Boolean networks. We only have to replace x^1 by $\{x^1, u\}$ to use the aforementioned technique.

7. Concluding remarks

This paper first considered the bi-decomposition of MVL functions. Necessary and sufficient conditions for both disjoint and non-disjoint cases were obtained. The conditions are easily verifiable and they provide a natural way to construct the decompositions. Then, as a particular bi-decomposition, a global Implicit Function Theorem of k-valued functions was obtained, which is necessary and sufficient. Finally, as an application, the normalization of D–A Boolean networks was considered.

The following are some final comments:

(i) Throughout this paper we assume that the concerned partitions are well ordered. In fact, the technique developed in this paper can also be used for arbitrary partition, which is briefly described here.

A matrix $W_{[m,n]}$ called *swap matrix* was defined in Cheng et al. (2012), which satisfies that, for any two column vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$,

$$W_{[m,n]}xy = yx. \tag{45}$$

Now assume that the function (1) is given with its algebraic form as

$$f(x_1, \ldots, x_n) = M_f \ltimes_{i=1}^{n} x_i. \tag{46}$$

Assume that f is bi-decomposable with respect to the order $\left(x_{i_1}, x_{i_2}, \ldots, x_{i_n}\right)$. The order can be seen as an element σ of the permutation group $\mathcal{S}_n$, that is, $\sigma : (1, 2, \ldots, n) \to (i_1, i_2, \ldots, i_n)$. Then we can construct a matrix W_σ such that

$$f(x_1, \ldots, x_n) = \tilde{M}_f \ltimes_{j=1}^{n} x_{i_j}, \tag{47}$$

where $\tilde{M}_f = M_f W_\sigma$. Using the methods developed in this paper to (47), we can check whether the decomposition is possible under this new order.

Next, we show how to construct W_σ. Define

$$W_1 := W_{[k_{i_1}, \prod_{j=1}^{i_1-1} k_j]}.$$

Using (45), we have

$$f(x_1, \ldots, x_n) = M_f W_1 x_{i_1} x_1 \cdots x_{i_1-1} x_{i_1+1} \cdots x_n.$$

Similarly, we can construct W_2 to move x_{i_2} to the second place. Keep going like this, we finally have

$$f(x_1, \ldots, x_n) = M_f W_1 x_{i_1} W_2 x_{i_2} \cdots W_{n-1} x_{i_{n-1}} x_{i_n}.$$

Using (5), we have $f(x_1, \ldots, x_n) = M_f W_1 \left(I_{k_{i_1}} \otimes W_2\right) \cdots \left(I_{k_{i_1}+\cdots+k_{i_{n-2}}} \otimes W_{n-1}\right) \ltimes_{j=1}^{n} x_{i_j}$. That is,

$$W_\sigma = W_1 \left(I_{k_{i_1}} \otimes W_2\right) \left(I_{k_{i_1}+k_{i_2}} \otimes W_3\right) \cdots \left(I_{k_{i_1}+\cdots+k_{i_{n-2}}} \otimes W_{n-1}\right).$$

(ii) Compared with previous results which only considered the Implicit Function Theorem of Boolean functions (Bazso, 2000), ours is a global one and is necessary and sufficient. From its application to D–A Boolean networks, one sees easily that the global k-valued IFT is particularly useful for logical functions. The method used there can also be extended to general D–A logical (control) networks.

(iii) Developing numerical algorithms for bi-decomposition is extremely important for practical use. But it is not the target of this paper. We leave it for further study.

References

Ashenhurst, R. L. (1957). The decomposition of switching functionis. *Annals of the Computation Laboratory of Harvard University*, *29*, 74–116.

Bazso, F. (2000). Derivation of vector-valued Boolean functions. *Acta Mathematica Hungarica*, *87*(3), 197–203.

Brayton, R.K., & Khatri, S.P. (1999). Multi-valued logic synthesis. In *Proceeding of the 12th international conference on VLSI design* (pp. 196–205). Goa, India.

Cheng, D. (2009). Input-state approach to Boolean networks. *IEEE Transactions on Neural Networks*, *20*(3), 512–521.

Cheng, D. (2011). Disturbance decoupling of Boolean control networks. *IEEE Transactions on Automatic Control*, *56*(1), 2–10.

Cheng, D., & Qi, H. (2009). Controllability and observability of Boolean control networks. *Automatica*, *45*(7), 1659–1667.

Cheng, D., & Qi, H. (2010a). A linear representation of dynamics of Boolean networks. *IEEE Transactions on Automatic Control*, *55*(10), 2251–2258.

Cheng, D., & Qi, H. (2010b). State-space analysis of Boolean networks. *IEEE Transactions on Neural Networks*, *21*(4), 584–594.

Cheng, D., Qi, H., & Li, Z. (2011). Analysis and control of Boolean networks. In *A semi-tensor product approach*. London: Springer.

Cheng, D., Qi, H., & Zhao, Y. (2012). *An introduction to semi-tensor product of matrices and its applications*. Singapore: World Scientific.

Curtis, H. A. (1962). *A new approach to the design of switching circuits*. Van Nostrand: Princeton.

Davio, M., Deschamps, J. P., & Thayse, A. (1978). *Discrete and switching functions*. New York: McGraw-Hill.

Didier, G., Remy, E., & Chaouiya, C. (2011). Mapping multivalued onto Boolean dynamics. *Journal of Theoretical Biology*, *270*(1), 177–184.

Files, C., & Perkowski, M. (1998). Multi-valued functional decomposition as a machine learning method. In *Proceedings of the the 28th international symposium on multiple-valued logic* (pp. 173–178). Fukuoka, Japan.

Hachtel, G. D., & Somenzi, F. (2002). *Logic synthesis and verification algorithms*. Boston: Kluwer Academic Publishers.

Hurst, S. L. (1984). Multiple-valued logic its status and its future. *IEEE Transctions on Computers*, *33*(12), 1160–1179.

Karp, R. (1963). Functional decomposition and switching circuit design. *Journal of SIAM*, *11*(2), 291–335.

Lang, C., & Steinbach, B. (2003). Bi-decomposition of function sets in multiple-valued logic for circuit design and data mining. *Artificial Intelligence Review*, *20*, 233–267.

Laschov, D., & Margaliot, M. (2011). A maximum principle for single-input Boolean control networks. *IEEE Transactions on Automatic Control*, *56*(4), 913–917.

Li, F., & Sun, J. (2011). Controllability of Boolean control networks with time delay in states. *Automatica*, *47*(3), 603–607.

Miller, D. M., & Thornton, M. A. (2007). *Multiple valued logic: concepts and representations*. San Rafael: Morgan and Claypool Publishers.

Mishchenko, A., & Brayton, R. (2002). A Boolean paradigm in multi-valued logic synthesis. In *International workshop on logic and synthesis* (pp. 173–177). New Orleans, USA.

Mishchenko, A., Steinbach, B., & Perkowski, M. (2001). An algorithm for bi-decomposition of logic functions. In *Proceedings of the 38th annual design automation conference* (pp. 103–108). Las Vegas.

Posthoff, C., & Steinbach, B. (2004). *Logic functions and equations: binary models for computer science*. Dordrecht: Springer.

Sasao, T. (1999). *Switching theory for logic synthesis*. Boston: Kluwer Academic Publishers.

Sasao, T., & Butler, J.T. (1989). On Bi-decompositions of logic functions. In *International workshop on logic and synthesis* (pp. 1–6). Lake Tahoe, USA.

Vykhovanets, V. S. (2006). Algebraic decomposition of discrete functions. *Automation and Remote Control*, *67*(3), 361–392.

Waliuzzaman, K. M., & Vranesic, Z. G. (1970). On decomposition of multi-valued switching functions. *The Computer Journal*, *13*(4), 359–362.

Zhao, Y., Li, Z., & Cheng, D. (2011). Optimal control of logical control networks. *IEEE Transactions on Automatic Control*, *56*(8), 1766–1776.

Zupan, B., Bohanec, M., Demsar, J., & Bratko, I. (1998). Feature transformation by function decomposition. *IEEE Transactions on Intelligent Systems*, *13*(2), 38–43.

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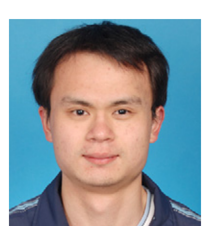

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