



Brief Paper

Constrained control of input–output linearizable systems using control sharing barrier functions[☆]

Xiangru Xu

Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI, USA

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ABSTRACT

Control barrier functions (CBFs) have been used as an effective tool for designing a family of controls that ensures the forward invariance of a set. When multiple CBFs are present, it is important that the set of controls satisfying all the barrier conditions is non-empty. In this paper, we investigate such a control-sharing property for multiple CBFs and provide sufficient and necessary conditions for the property to hold. Based on that, we study the tracking control design problem of an input–output linearizable system with multiple time-varying output constraints, where the output constraints are encoded as CBFs and the barrier conditions are expressed as hard constraints in a quadratic program (QP) whose feasibility is guaranteed by the control-sharing property of the CBFs. With the controller generated from the QP, the output constraints are always satisfied and the tracking objective is achieved when it is not conflicting with the constraints. The effectiveness of our control design method is illustrated by two examples.

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1. Introduction

First introduced in optimization, barrier functions (also known as barrier certificates) are now used as an important tool for the verification of nonlinear systems and hybrid systems (Prajna & Jadbabaie, 2004; Prajna, Jadbabaie, & Pappas, 2007; Wisniewski & Sloth, 2016). Using Lyapunov-like conditions, barrier functions can provably establish safety or eventuality properties of dynamical systems without the difficult task of computing the system's reachable set. The extension of barrier functions to a control system results in control barrier functions (CBFs), which, in some sense, parallels the extension of Lyapunov functions to Control Lyapunov function (CLFs). A family of controls ensuring the forward invariance of a set is established by the barrier condition, which can be used for the control synthesis of systems with state constraints or safety specifications (Ames, Grizzle, & Tabuada, 2014; Panagou, Stipanović, & Voulgaris, 2016; Tee, Ge, & Tay, 2009; Wieland & Allgöwer, 2007).

Depending on the values of a CBF on the associated set, two types of (control) barrier functions are commonly used in literature: one goes to infinity on the set boundary (Ames et al., 2014; Jin & Xu, 2013; Tee et al., 2009), while the other vanishes on the set

boundary (Romdlony & Jayawardhana, 2016; Wolff & Buss, 2005; Xu, Tabuada, Ames, & Grizzle, 2015). The former type of CBFs is only defined inside the given set whose boundary cannot be crossed; for example, the reciprocal CBF in Ames et al. (2014), the barrier Lyapunov function (BLF) in Ngo, Mahony, and Jiang (2005) and Tee et al. (2009), and several of its extensions such as the tan-type BLF (Jin, 2017) and the integral BLF (He, Sun, & Ge, 2015). The latter type of CBFs is defined in the whole state space, but the barrier condition ensures that the trajectory of the system will stay inside the set once starting there. Related works belonging to this type include the invariance control (Kimmel & Hirche, 2015; Kimmel, Jahne, & Hirche, 2016; Wolff & Buss, 2005), the control Lyapunov barrier function (Romdlony & Jayawardhana, 2016), and the zeroing CBF (Xu et al., 2015), among others.

Various kinds of barrier conditions have been proposed in literature. A widely used barrier condition for a CBF B is $\dot{B} \leq 0$ (or $\dot{B} \geq 0$ depending on the context), which implies that all the sublevel sets of B are invariant (Prajna et al., 2007; Romdlony & Jayawardhana, 2016; Tee et al., 2009; Wieland & Allgöwer, 2007). Another barrier condition is that given in the invariance control framework, where the higher order derivative condition of a so-called invariance function is implemented such that the function has negative values inside the set. In a recent paper Ames et al. (2014), the barrier condition $\dot{B} \leq 0$ was modified by allowing B to grow when it is far away from the boundary of the set and stop growing when it approaches the boundary. Such a condition enlarges the set of controls that can guarantee the invariance of a given set. CBFs under such a condition are combined with CLFs, which represent

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E-mail address: xuxiangr@umich.edu.

the performance objectives, in a quadratic program (QP), such that a min-norm control law is generated via real-time optimizations. This idea was further extended in papers such as Ames, Xu, Grizzle, and Tabuada (2017), Nguyen and Sreenath (2016) and Xu et al. (2015), and applied to safety-critical systems (Ames et al., 2017), multi-agent systems (Wang, Ames, & Egerstedt, 2016) and bipedal robots (Hsu, Xu, & Ames, 2015).

When multiple state constraints are presented and each constraint is expressed as a CBF, it is important to ensure that all the barrier conditions can be satisfied simultaneously, that is, the set of controls satisfying all the barrier conditions is non-empty. Particularly, for the QP-based framework proposed in Ames et al. (2014, 2017) and Xu et al. (2015), simultaneous satisfaction of all the barrier conditions is needed to guarantee the feasibility of the QP. Such a shared-control problem has been investigated for CLFs in Andrieu and Prieur (2010), Grammatico, Blanchini, and Andrea (2014) and shown to be hard to solve in general; for instance, it was shown in Grammatico et al. (2014) that two convex CLFs do not necessarily have a common control even for linear time-invariant systems when the dimension of the system is greater than 2.

In this paper, we study the control-sharing property of multiple high order CBFs. Roughly speaking, CBFs are said to have the control-sharing property if for any state, there exists a common control such that the barrier conditions are satisfied simultaneously. Sufficient and necessary conditions for the control-sharing property to hold are given by assuming the CBFs have a well-defined, global relative degree. Based on that, we investigate the tracking control problem for input–output linearizable systems with multiple time-varying output constraints, where each constraint is expressed as a CBF. Sufficient conditions for such CBFs to have the control-sharing property are given. The barrier conditions are expressed as hard constraints in a QP, where the objective function is to minimize the distance between the generated control and a nominal tracking control law. Because of the control-sharing property of the CBFs, the QP is guaranteed to be feasible. Furthermore, the output constraints are always satisfied and the tracking objective is achieved when it is not conflicting with the constraints. Our control design method has several advantages over existing ones, such as the output constraints and the nominal tracking controller can be designed separately, the reference trajectory does not need to be restricted inside the constraint region, and the initial output can be outside the constraint region. Two examples taken from literature are also provided to show the effectiveness of the proposed control design method.

A preliminary version of this work was presented in the conference publication Xu (2016). The present paper is different from Xu (2016) in the following important ways: the input–output linearizable system (instead of the strict-feedback system in Xu, 2016) is considered; a key theorem in Xu (2016) is generalized from the sufficient condition to the sufficient and necessary conditions; the two CBFs case is generalized to the multiple CBFs case. The remainder of the paper is organized as follows. In Section 2, the notion of time-varying control barrier function and the control-sharing property are introduced first, then sufficient and necessary conditions for the control-sharing property to hold are given. In Section 3, the tracking control problem for input–output linearizable systems with multiple output constraints is investigated, where two examples are also provided for illustrative purposes. Finally, some conclusion remarks are given in Section 4.

2. Control-sharing barrier functions

In this section, we first provide a lemma for ensuring non-negativeness of a function through a high order derivative condition, and then introduce the notion of high order, time-varying

CBFs. After that, we define the control-sharing property of multiple CBFs and give sufficient and necessary conditions for such a property to hold.

2.1. Control barrier function

Consider a time-varying system

$$\dot{x} = f(t, x), \quad (1)$$

with $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ piecewise continuous in t and locally Lipschitz in x . For any initial condition $x(0)$ at $t = 0$, there exists a maximal time interval $I(x(0))$ such that $x(t)$ is the unique solution to (1). For simplicity, we assume that the system (1) is forward complete, that is, $I(x(0)) = [0, \infty)$.

Given a smooth function $h(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, its first order derivative along the solution of (1) is $\dot{h}(t, x) = \frac{dh(t, x)}{dt} = \frac{\partial h(t, x)}{\partial t} + \frac{\partial h(t, x)}{\partial x} f(t, x)$. The i th ($i \geq 2$) order derivative of $h(t, x)$ is computed recursively and denoted as $h^{(i)}(t, x)$. In what follows, we will also use $h^{(i)}$ for $h^{(i)}(t, x)$ when no confusion occurs.

Now suppose that $h(t, x)$ is a C^r function for some positive integer $r \geq 1$ and satisfies the following inequality:

$$h^{(r)} + a_1 h^{(r-1)} + \dots + a_{r-1} h^{(1)} + a_r h \geq 0, \quad (2)$$

where $a_1, \dots, a_r \in \mathbb{R}$ are a set of real numbers such that the roots of the polynomial

$$p_0^r(\lambda) = \lambda^r + a_1 \lambda^{r-1} + \dots + a_{r-1} \lambda + a_r \quad (3)$$

are real numbers $-\lambda_1, \dots, -\lambda_r$ with $\lambda_i > 0$ ($1 \leq i \leq r$). To explore the condition under which $h(t, x)$ is non-negative for $t \geq 0$, we define

$$s_0(t, x) = h(t, x), \quad s_k = \left(\frac{d}{dt} + \lambda_k \right) s_{k-1}, \quad 1 \leq k \leq r. \quad (4)$$

It is clear that (2) is equivalent to $s_r(t, x) \geq 0$. Denote $s_k(0, x(0))$ by $s_k(0)$ for short where $k = 0, 1, \dots, r$. Then, we have the following lemma.

Lemma 1. Given a C^r ($r \geq 1$) function $h(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and a set of real numbers $a_1, \dots, a_r \in \mathbb{R}$ such that $p_0^r(\lambda)$ shown in (3) has roots $-\lambda_1, \dots, -\lambda_r$ where $\lambda_1, \dots, \lambda_r > 0$, if s_i defined in (4) satisfy $s_i(0) \geq 0$ for $i = 0, 1, \dots, r-1$, then $h(t, x) \geq 0$ for any $t \geq 0$.

Proof. It is clear that inequality (2) is equivalent to $\frac{d}{dt}(e^{\lambda_r t} s_{r-1}(t, x(t))) \geq 0$, which results in $s_{r-1}(t, x(t)) \geq s_{r-1}(0)e^{-\lambda_r t}$ by integrating both sides on $[0, t]$. Since $s_{r-1} = (\frac{d}{dt} + \lambda_{r-1})s_{r-2}$, we have $\frac{d}{dt}(e^{\lambda_{r-1} t} s_{r-2}(t, x(t))) \geq s_{r-1}(0)e^{(\lambda_{r-1} - \lambda_r)t}$. Integrating both sides of this inequality on $[0, t]$ results in $s_{r-2}(t, x(t)) \geq s_{r-1}(0)e^{-\lambda_{r-1} t} \int_0^t e^{(\lambda_{r-1} - \lambda_r)\tau_1} d\tau_1 + s_{r-2}(0)e^{-\lambda_{r-1} t}$. Continuing this process, we have

$$s_0(t, x(t)) \geq s_0(0)e^{-\lambda_1 t} + \sum_{k=1}^{r-1} [s_k(0)e^{-\lambda_1 t} \int_0^t e^{(\lambda_1 - \lambda_2)\tau_k} \int_0^{\tau_k} e^{(\lambda_2 - \lambda_3)\tau_{k-1}} \dots \int_0^{\tau_2} e^{(\lambda_k - \lambda_{k+1})\tau_1} d\tau_1 \dots d\tau_{k-1} d\tau_k]. \quad (5)$$

For $k = 1, \dots, r-1$, since $\lambda_i > 0$, it is easy to check that $e^{-\lambda_1 t} \int_0^t e^{(\lambda_1 - \lambda_2)\tau_k} \int_0^{\tau_k} e^{(\lambda_2 - \lambda_3)\tau_{k-1}} \dots \int_0^{\tau_2} e^{(\lambda_k - \lambda_{k+1})\tau_1} d\tau_1 \dots d\tau_{k-1} d\tau_k$ is positive, finite and approaches 0 as $t \rightarrow \infty$. Since $s_i(0) \geq 0$ for $i = 0, 1, \dots, r-1$, the right-hand side of (5) is non-negative, finite and approaches 0 as $t \rightarrow \infty$. Therefore, $h(t, x) \geq 0$ for any $t \geq 0$, which completes the proof. \square

Remark 1. The conventional comparison lemma cannot be applied to the high order inequality (2) directly (Khalil, 2002). In Gunderson (1971), Gunderson considered the high order differential

inequality $v^{(m)} \leq g(t, v, v^{(1)}, \dots, v^{(m-1)})$ and compared its solution with the co-system $u^{(m)} = g(t, u, u^{(1)}, \dots, u^{(m-1)})$, under the assumption that the map $g(\cdot)$ has some non-decreasing property W^* . However, if $g(\cdot)$ is a linear time invariant equation such as that in (2), the W^* property does not hold. In Meigoli, Kamaledin, and Nikravesh (2009, 2012), Meigoli proposed high order derivative conditions for Lyapunov function candidates such that some stability results in the sense of Lyapunov can be obtained. A “counter-example” was given in Example 2 of Meigoli et al. (2009) showing that if the characteristic equation (3) has complex roots, then the solution of (2) may be greater than the solution of its co-system at some time.

Now consider a time-varying affine control system

$$\dot{x} = f(t, x) + g(t, x)u, \quad (6)$$

where $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are piecewise continuous in t and locally Lipschitz in x . Given a sufficiently smooth time-varying function $h(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, we define the modified Lie derivative of $h(t, x)$ along f as $\bar{L}_f^i h := (\frac{\partial}{\partial t} + L_f)^i h$ where i is a non-negative integer (see Palanki and Kravaris, 1997 for more details). We formally define the high order, time-varying CBFs, which generalizes the zeroing CBF of order 1 proposed in Xu et al. (2015), as follows.

Definition 1. Given control system (6), a C^r function $h(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ with a relative degree r is called a (zeroing) control barrier function (of order r) if there exists a column vector $\mathbf{a} = (a_1, \dots, a_r)' \in \mathbb{R}^r$ such that for all $x \in \mathbb{R}^n$, $t \geq 0$,

$$\sup_{u \in U} [L_g \bar{L}_f^{r-1} h(t, x)u + \bar{L}_f^r h(t, x) + \mathbf{a}' \xi(t, x)] \geq 0, \quad (7)$$

where $\xi(t, x) = (\bar{L}_f^{r-1} h, \bar{L}_f^{r-2} h, \dots, h)' \in \mathbb{R}^r$, and the roots of $p_0^r(\lambda)$ shown in (3) are all negative.

For the rest of the paper, we assume that there is no constraint on the input u (i.e., $U = \mathbb{R}$). Define set $\mathcal{K}(t, x) := \{u \in \mathbb{R} | L_g \bar{L}_f^{r-1} h(t, x)u + \bar{L}_f^r h(t, x) + \mathbf{a}' \xi(t, x) \geq 0\}$ for $t \geq 0$, $x \in \mathbb{R}^n$. Then we have the following result that ensures the non-negativeness of h .

Proposition 1. Given control system (6), if (i) $h(t, x)$ with a relative degree r is a CBF such that (7) holds and the roots of $p_0^r(\lambda)$ are $-\lambda_1, \dots, -\lambda_r < 0$, (ii) s_i defined in (4) satisfy $s_i(0) \geq 0$ for $i = 0, 1, \dots, r-1$, then any controller $u(t, x) \in \mathcal{K}(t, x)$ that is Lipschitz in x will render $h(t, x) \geq 0$ for any $t \geq 0$.

Proposition 1 can be proved immediately by Lemma 1 (Blanchini & Miani, 2015; Xu et al., 2015). The set $\mathcal{K}(t, x)$ provides a family of controls that guarantees the controlled invariance of a time-varying set $\mathcal{C}_t := \{x \in \mathbb{R}^n | h(t, x) \geq 0\}$, such that $x(t) \in \mathcal{C}_t$ for all $t \geq 0$. Compared with the reciprocal-type CBFs proposed in Ames et al. (2014) and the BLFs proposed in Tee et al. (2009), which both assume that the barrier function takes the value of infinity on the set boundary $\partial \mathcal{C}_t$, the CBF defined above takes zero values on the set boundary and allows h to change sign, which is a nice property that will show its advantage later.

Remark 2. High order barrier conditions have been explored in literature; for example, Hsu et al. (2015) studied the reciprocal-type CBFs using a backstepping-like method, Nguyen and Sreenath (2016) investigated the “exponential CBFs” that is similar to ours and Wolff and Buss (2005) studied the invariance condition of a set using high order derivatives of a so-called invariance function. Furthermore, it deserves mentioning that ideas similar to the CBFs also appeared in the nonovershooting control of nonlinear systems (Krstic & Bement, 2006; Zhu, 2013).

2.2. Control-sharing property

In this subsection, we define the control-sharing property for multiple CBFs and provide sufficient and necessary conditions for such a property to hold.

Given control system (6) and $q(q \geq 2)$ CBFs $h_i(t, x)$, $i = 1, \dots, q$, a natural question to ask is whether there exists a common (or shared) control $u(t, x)$ such that the barrier conditions (7) for h_i are satisfied simultaneously. We formally define this property as the *control-sharing property* as follows.

Definition 2. Consider control system (6) and $q(q \geq 2)$ CBFs $h_i(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, q$, where h_i has a well-defined (global, uniform) relative degree r_i , that is, $L_g \bar{L}_f^{r_i-1} h_i(t, x) \neq 0$, $L_g \bar{L}_f^i h_i(t, x) = 0$, $i = 0, 1, \dots, r_i - 2$, for any $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. Suppose that (7) holds with $\xi_i(t, x) = (\bar{L}_f^{r_i-1} h_i, \bar{L}_f^{r_i-2} h_i, \dots, h_i)'$ for some $\mathbf{a}_i = (a_1^i, \dots, a_{r_i}^i)'$ where the roots of $p_0^{r_i}(\lambda)$ are all negative. The CBFs h_1, \dots, h_q are said to have the *control-sharing property* if for any $t \geq 0$, $x \in \mathbb{R}^n$, there exists control $u \in \mathbb{R}$ such that the following condition holds for any $i = 1, \dots, q$:

$$L_g \bar{L}_f^{r_i-1} h_i(t, x)u + \bar{L}_f^{r_i} h_i(t, x) + \mathbf{a}_i' \xi_i(t, x) \geq 0. \quad (8)$$

The CBFs h_1, \dots, h_q are said to be *control-sharing barrier functions* (CSBFs) if they have the control-sharing property. Define sets $\mathcal{K}_i(t, x) := \{u \in \mathbb{R} | L_g \bar{L}_f^{r_i-1} h_i(t, x)u + \bar{L}_f^{r_i} h_i(t, x) + \mathbf{a}_i' \xi_i(t, x) \geq 0\}$ for $i = 1, \dots, q$. Then h_1, \dots, h_q are CSBFs implies $\bigcap_{i=1}^q \mathcal{K}_i(t, x) \neq \emptyset$ for all $t \geq 0$, $x \in \mathbb{R}^n$.

The next theorem provides sufficient and necessary conditions for two functions h_1, h_2 to be CSBFs.

Theorem 1. Consider control system (6) and two CBFs $h_1(t, x)$, $h_2(t, x)$ that are assumed to have well-defined (global, uniform) relative degrees $r_1, r_2 \geq 1$, respectively.

(i) If $\text{sgn}(L_g \bar{L}_f^{r_1-1} h_1(t, x)) \text{sgn}(L_g \bar{L}_f^{r_2-1} h_2(t, x)) = 1$, then h_1, h_2 are CSBFs.

(ii) If $L_g \bar{L}_f^{r_1-1} h_1(t, x) > 0$, $L_g \bar{L}_f^{r_2-1} h_2(t, x) < 0$, then h_1, h_2 are CSBFs if and only if for any $t \geq 0$, $x \in \mathbb{R}^n$,

$$L_g \bar{L}_f^{r_1-1} h_1(t, x)[\bar{L}_f^{r_2} h_2(t, x) + \mathbf{a}_2' \xi_2(t, x)] \geq L_g \bar{L}_f^{r_2-1} h_2(t, x)[\bar{L}_f^{r_1} h_1(t, x) + \mathbf{a}_1' \xi_1(t, x)]. \quad (9)$$

(iii) If $L_g \bar{L}_f^{r_1-1} h_1(t, x) < 0$, $L_g \bar{L}_f^{r_2-1} h_2(t, x) > 0$, then h_1, h_2 are CSBFs if and only if for any $t \geq 0$, $x \in \mathbb{R}^n$,

$$L_g \bar{L}_f^{r_1-1} h_1(t, x)[\bar{L}_f^{r_2} h_2(t, x) + \mathbf{a}_2' \xi_2(t, x)] \leq L_g \bar{L}_f^{r_2-1} h_2(t, x)[\bar{L}_f^{r_1} h_1(t, x) + \mathbf{a}_1' \xi_1(t, x)]. \quad (10)$$

Proof. (i) For any $t \geq 0$, $x \in \mathbb{R}^n$, if $L_g \bar{L}_f^{r_1-1} h_1(t, x)$ and $L_g \bar{L}_f^{r_2-1} h_2(t, x)$ have the same sign, then it is clear that there exists u such that (8) for $k = 1, 2$ can be satisfied simultaneously. (ii) In this case,

(8) for $k = 1$ is equivalent to $u \geq \frac{-\bar{L}_f^{r_1} h_1(t, x) - \mathbf{a}_1' \xi_1(t, x)}{L_g \bar{L}_f^{r_1-1} h_1(t, x)}$, and (8) for

$k = 2$ is equivalent to $u \leq \frac{\bar{L}_f^{r_2} h_2(t, x) + \mathbf{a}_2' \xi_2(t, x)}{-L_g \bar{L}_f^{r_2-1} h_2(t, x)}$. Therefore, h_1, h_2 are

CSBFs if and only if for any $t \geq 0$, $x \in \mathbb{R}^n$, $\frac{-\bar{L}_f^{r_1} h_1(t, x) - \mathbf{a}_1' \xi_1(t, x)}{L_g \bar{L}_f^{r_1-1} h_1(t, x)} \leq$

$\frac{\bar{L}_f^{r_2} h_2(t, x) + \mathbf{a}_2' \xi_2(t, x)}{-L_g \bar{L}_f^{r_2-1} h_2(t, x)}$ holds, which is equivalent to condition (9) by mul-

tiplying $(L_g \bar{L}_f^{r_1-1} h_1)(-L_g \bar{L}_f^{r_2-1} h_2)$ on both sides. (iii) Similar to case (ii). \square

Sufficient and necessary conditions for the control-sharing property of multiple CBFs can be given similar to Theorem 1, which is omitted here due to the space limitation.

3. Tracking control of input–output linearizable systems with output constraints

In this section, we consider the tracking control design problem of input–output linearizable systems with multiple time-varying output constraints, where the reference trajectory may conflict with the constraints. To solve the problem, we express each constraint as a CBF and provide conditions for them to be CSBFs. The control law is obtained by solving a QP with the barrier conditions as hard constraints, such that the output constraints are always satisfied and the tracking objective is achieved when it does not conflict with the constraints. The QP is always feasible because of the control-sharing property of CBFs.

Consider a SISO affine control system as follows:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u, \\ y &= h(x),\end{aligned}\quad (11)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are functions locally Lipschitz in x and $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is a sufficiently smooth function.

Suppose that $h(x)$ has a well-defined, global relative degree r ($1 \leq r \leq n$) on \mathbb{R}^n , that is, for all $x \in \mathbb{R}^n$, $L_g L_f^i h(x) = 0$, $i = 0, 1, \dots, r-2$, and $L_g L_f^{r-1} h(x) \neq 0$. We assume further that the sign of $L_g L_f^{r-1} h$ is positive.

Assumption 1. For all $x \in \mathbb{R}^n$, $L_g L_f^{r-1} h(x) > 0$.

When its relative degree $r < n$, system (11) can be transformed into a normal form with internal dynamics $\dot{\eta} = f_0(\eta, \xi)$ via the standard input–output linearization (Khalil, 2002). We pose the following assumption on the internal dynamics.

Assumption 2. The internal dynamics $\dot{\eta} = f_0(\eta, \xi)$ of (11) is bounded-input-bounded-state stable.

Consider a (bounded) reference trajectory $y_d(t): \mathbb{R}_+ \rightarrow \mathbb{R}$, and two C^r functions $\bar{y}(t): \mathbb{R}_+ \rightarrow \mathbb{R}$, $\underline{y}(t): \mathbb{R}_+ \rightarrow \mathbb{R}$ where $\bar{y}(t) > \underline{y}(t)$ and $\bar{y}^{(i)}, \underline{y}^{(i)}$ are bounded for $i = 0, \dots, r$, $t \geq 0$. Suppose that the initial condition $x(0) \in X_0$ where X_0 is a compact set such that $y(0) < \underline{y}(0) < \bar{y}(0)$. The control objective is to design a feedback controller such that (i) the output y satisfies the constraint:

$$\underline{y}(t) \leq y(t) \leq \bar{y}(t), \quad \forall t \geq 0, \quad (12)$$

and (ii) the output y tracks $y_d(t)$ as close as possible.

Note that no restriction is posed on the relation between $y_d(t)$ and $\bar{y}(t)$, $\underline{y}(t)$. Therefore, the tracking objective above can be interpreted as follows: $y(t) \rightarrow y_d(t)$ if $y(t) \leq y_d(t) \leq \bar{y}(t)$, $y(t) \rightarrow \bar{y}(t)$ if $y_d(t) > \bar{y}(t)$, and $y(t) \rightarrow \underline{y}(t)$ if $y_d(t) < \underline{y}(t)$. In other words, while ensuring the bounding constraint (12) always satisfied, the tracking performance is compromised when the reference trajectory conflicts with the constraint.

Define two CBF candidates $h_1(t, x)$, $h_2(t, x)$ as

$$h_1(t, x) = y(x(t)) - \underline{y}(t), \quad (13)$$

$$h_2(t, x) = \bar{y}(t) - y(x(t)). \quad (14)$$

Then, the output constraint (12) is equivalent to requiring $h_1(t, x) \geq 0$ and $h_2(t, x) \geq 0$ for all $t \geq 0$. The following theorem provides sufficient conditions for h_1 , h_2 to be CSBFs and $h_1(t, x)$, $h_2(t, x) \geq 0$ for all $t \geq 0$.

Theorem 2. Consider nonlinear system (11) where y has a well-defined, global relative degree r ($1 \leq r \leq n$) on \mathbb{R}^n , and two C^r functions $\bar{y}(t)$, $\underline{y}(t)$ where $\bar{y}(t) > \underline{y}(t)$ for all $t \geq 0$ and $\bar{y}^{(i)}, \underline{y}^{(i)}$ are bounded for $i = 0, \dots, r$. Suppose that Assumptions 1 and 2 hold, and $x(0) \in X_0$ where X_0 is a compact set such that $\underline{y}(0) < y(0) < \bar{y}(0)$.

Then there exist $a_1, \dots, a_r \in \mathbb{R}$ such that (i) the roots of $p_0^r(\lambda)$ defined in (3) are all negative, (ii) $s_k^1(0)$ defined in (4) with respect to h_1 in (13) and $s_k^2(0)$ with respect to h_2 in (14) are all non-negative for $k = 0, 1, \dots, r-1$. Furthermore, if

$$\sum_{j=0}^r a_{r-j}(\bar{y}^{(j)}(t) - \underline{y}^{(j)}(t)) \geq 0, \quad \forall t \geq 0, \quad (15)$$

holds with such a_1, \dots, a_r and $a_0 := 1$, then h_1 and h_2 are CSBFs, and $h_1(t, x)$, $h_2(t, x) \geq 0$ for $t \geq 0$.

Proof. Since y has a well-defined relative degree r on \mathbb{R}^n , functions h_1, h_2 also have a well-defined, global relative degree r . Because h_1, h_2 are C^r functions and X_0 is compact, $|h_1^{(j)}(0, x(0))|$, $|h_2^{(j)}(0, x(0))|$ are bounded for $j = 0, \dots, r-1$. Recalling (4), because $h_1(0), h_2(0) > 0$, there exist $\lambda_1, \dots, \lambda_r > 0$ such that $s_1^1(0), s_2^1(0), \dots, s_{r-1}^1(0) > 0$ for $i = 1, 2$. Therefore, there exist $a_1, \dots, a_r \in \mathbb{R}$ such that the roots of $p_0^r(\lambda)$ are $-\lambda_1, \dots, -\lambda_r < 0$, and $s_k^i(0) > 0$ for $i = 1, 2$ and $k = 0, 1, \dots, r-1$.

With such a_0, a_1, \dots, a_r , if h_i ($i = 1, 2$) satisfies the following condition:

$$L_g \bar{L}_f^{r-1} h_i u + \sum_{j=0}^r a_{r-j} \bar{L}_f^j h_i \geq 0, \quad (16)$$

then, by case (ii) of Theorem 1, h_1, h_2 are CSBFs if and only if $L_g \bar{L}_f^{r-1} y[-\sum_{j=0}^r a_{r-j} \bar{L}_f^j y + \sum_{j=0}^r a_{r-j} \bar{y}^{(j)}] \geq -L_g \bar{L}_f^{r-1} y[\sum_{j=0}^r a_{r-j} \bar{L}_f^j y - \sum_{j=0}^r a_{r-j} \bar{y}^{(j)}]$ because $L_g \bar{L}_f^{r-1} h_1 = L_g \bar{L}_f^{r-1} y > 0$, $L_g \bar{L}_f^{r-1} h_2 = -L_g \bar{L}_f^{r-1} y < 0$. Clearly, this inequality is implied by condition (15). Therefore, h_1, h_2 are CSBFs, and the conclusion $h_1(t, x) \geq 0$, $h_2(t, x) \geq 0$ follows by Lemma 1. \square

If a_1, \dots, a_r are chosen such that conditions of Theorem 2 are satisfied and h_1, h_2 are CSBFs, then, as discussed in Section 2.2, any Lipschitz-continuous controller $u(t, x) \in \mathcal{K}_1(t, x) \cap \mathcal{K}_2(t, x)$ guarantees that (12) holds.

To achieve the tracking objective, a nominal tracking control law \hat{u} can be designed without considering the output constraint, by transforming the tracking problem as an asymptotic stabilization problem using various standard approaches (Blanchini & Miani, 2000; Khalil, 2002), such as the backstepping control design when the system is in strict-feedback form (Krstic, Kokotovic, & Kanellakopoulos, 1995), or the output regulation when the reference trajectory is generated by certain classes of exosystems (Byrnes, Priscoli, & Isidori, 2012).

Among all the controls in the set $\mathcal{K}_1 \cap \mathcal{K}_2$, we choose the control u that minimizes $\|u - \hat{u}\|_2$ (Freeman & Kokotovic, 1996). This min-norm controller is obtained by solving the following convex quadratic program:

$$u^*(x) = \underset{u \in \mathbb{R}}{\operatorname{argmin}} \|u - \hat{u}\|_2 \quad (\text{CBF-QP})$$

$$\text{s.t. } \phi_1^0 u + \phi_1^1 \leq 0, \quad (\text{CBF1})$$

$$\phi_2^0 u + \phi_2^1 \leq 0, \quad (\text{CBF2})$$

where (CBF1)–(CBF2) correspond to condition (16) with $\phi_1^0 = -L_g \bar{L}_f^{r-1} y$, $\phi_1^1 = -\sum_{j=0}^r a_{r-j} \bar{L}_f^j y + \sum_{j=0}^r a_{r-j} \bar{y}^{(j)}$, $\phi_2^0 = L_g \bar{L}_f^{r-1} y$, $\phi_2^1 = \sum_{j=0}^r a_{r-j} \bar{L}_f^j y - \sum_{j=0}^r a_{r-j} \bar{y}^{(j)}$. Because h_1, h_2 are CSBFs, there always exists u such that (CBF1)–(CBF2) are satisfied simultaneously for any $t \geq 0$, $x \in \mathbb{R}^n$, which means that the QP (CBF-QP) is always feasible. While the control u ensures that the output constraint (12) is respected, it is also chosen to be close to \hat{u} as much as possible. Hence, $y(t)$ tracks $y_d(t)$ as close as possible. Particularly, $y(t) \rightarrow y_d(t)$ will be achieved when the constraints (CBF1)–(CBF2) are inactive.

One advantage of the proposed QP-based framework is that, the output constraint and the tracking objective are considered separately, which makes the control design procedure more flexible.

On one hand, the nominal controller in (CBF-QP) can be replaced by any existing controller, such as a legacy controller in some practical problem or a human operator in the “human-in-the-loop” mechanism (Jiang & Astolfi, 2016; Kimmel & Hirche, 2015). Therefore, our framework can be used as an “add-on” to another control law such that the safety specification can be ensured. On the other hand, because of the *separability* of the “safety constraint” and “control performance”, the assumption $y(t) < y_d(t) < \bar{y}(t)$ is not required, which was in fact used as a standing assumption in papers that deal with similar problems (Jiang & Astolfi, 2016; Jin & Xu, 2013; Tee et al., 2009).

Another advantage of the QP-based framework is that the boundary of the constraint region can be crossed by the output trajectory. For instance, when the initial output is outside the constraint region, or the output is steered out of the region due to unknown disturbances, it can enter the constraint region again (by tracking the reference trajectory) and stay inside it afterwards, provided that certain conditions hold. Specifically, if for some $T_0 > 0$, $s_k^i(T_0, x(T_0))$ become non-negative for all $k = 1, \dots, r$ and $i = 1, 2$, then $h_1(t, x), h_2(t, x) \geq 0$ for any $t \geq T_0$; if no such T exists, the right-hand side of (5) provides lower bounds for h_1, h_2 , which approach 0 as t goes to infinity. Note that crossing the constraint boundary is not possible in other related papers using the reciprocal-type barriers such as Jiang and Astolfi (2016), Jin and Xu (2013) and Tee et al. (2009), because the barrier functions there take the value of infinity on the set boundary.

Remark 3. Condition (9) (or (10), (15)) can be checked by posing it as an unconstrained minimization problem on variables t, x . When dynamics of the system (as well as the constraint functions) are polynomial or rational (or can be transformed into polynomial/rational after variable substitutions), the coefficients a_1, \dots, a_r satisfying (9) (or (10), (15)) can be found using the sum-of-squares (SOS) optimization (Parrilo, 2000). Note that the obtained coefficients also need to render all the roots of $p_0'(\lambda)$ to be negative.

We use the following Example 1 from Tee, Ren, and Ge (2011) to illustrate the effectiveness of our control design method.

Example 1. Consider a system described by

$$\begin{aligned}\dot{x}_1 &= 0.1x_1^2 + x_2, \\ \dot{x}_2 &= 0.1x_1x_2 - 0.2x_1 + (1 + x_1^2)u.\end{aligned}$$

The output of the system is $y = x_1$, the reference output is $y_d(t) = 0.5 \sin t$ and the constraint functions are $\bar{y}(t) = 0.6 + 0.1 \cos t$, $\underline{y}(t) = -0.5 + 0.4 \sin t$. This system is a strict-feedback system without internal dynamics, hence Assumption 2 holds; it has a global relative degree 2 since $L_g L_f y = 1 + x_1^2 > 0$ for all $x \in \mathbb{R}^2$, hence Assumption 1 holds. We suppose that the initial states satisfy $|x_1(0)| \leq 0.4$, $|x_2(0)| \leq 2.5$.

According to (13) and (14), we define $h_1(x, t) = x_1 - 0.4 \sin t + 0.5$ and $h_2(x, t) = 0.1 \cos t + 0.6 - x_1$. We use the SOS optimization to find $a_1, a_2 \in \mathbb{R}$ such that (15) holds. Specifically, we find $a_1, a_2 > 0$ such that $(\ddot{\bar{y}}(t) - \ddot{y}(t)) + a_1(\dot{\bar{y}}(t) - \dot{y}(t)) + a_2(\bar{y}(t) - y(t)) \geq 0$, where $\cos t$ and $\sin t$ are defined respectively as additional variables z_1, z_2 with the constraint $z_1^2 + z_2^2 = 1$. By the SOS, we obtain $a_1 = 32.54$, $a_2 = 43.34$, which implies that the roots of $p_0^2(\lambda)$ are $-\lambda_1 = -31.15$, $-\lambda_2 = -1.39$. One can verify that $(\ddot{\bar{y}}(t) - \ddot{y}(t)) + a_1(\dot{\bar{y}}(t) - \dot{y}(t)) + a_2(\bar{y}(t) - y(t)) = 47.67 - 20.19 \sin t - 8.78 \cos t \geq 25.65$, and $h_1^{(1)}(0) + \lambda_1 h_1(0) > 0$, $h_2^{(1)}(0) + \lambda_1 h_2(0) > 0$ for all $(x_1(0), x_2(0)) \in X_0$. Therefore, all the conditions of Theorem 2 are satisfied.

We obtain the nominal tracking control law \hat{u} via the backstepping approach, and generate the input u by solving the quadratic program (CBF-QP). The simulation result is shown in Fig. 1, where the method proposed in this paper is compared with that in Tee et

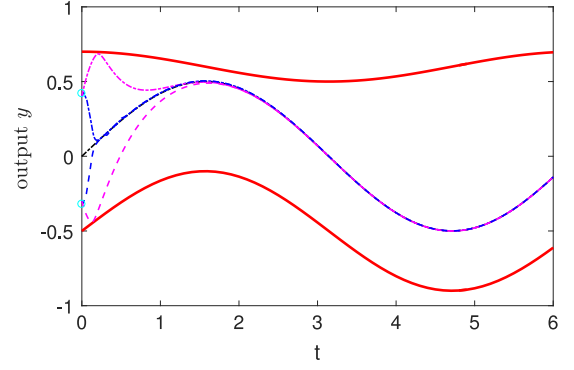


Fig. 1. Dash-dot black line: $y_d = 0.5 \sin t$; solid red lines: $y(t)$ and $\bar{y}(t)$; dash-dot and dash blue lines: the output using our method from the initial conditions (0.4, 2.5) and $(-0.3, -2)$, respectively; dash-dot and dash magenta lines: the output using the method in Tee et al. (2011) from the initial conditions (0.4, 2.5) and $(-0.3, -2)$, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

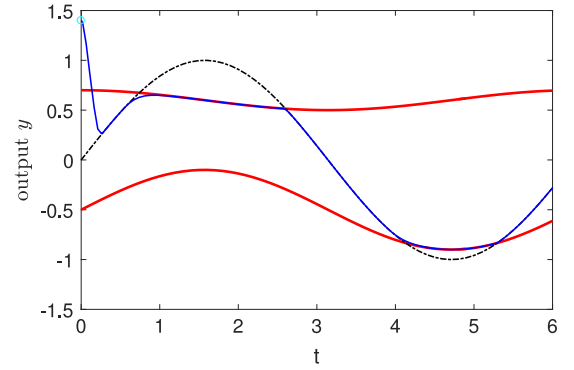


Fig. 2. Dash-dot black line: $y_d = \sin t$; red lines: $y(t)$ and $\bar{y}(t)$; blue line: the output trajectory using our method from the initial condition (1.4, 0). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

al. (2011). The reference trajectory y_d is depicted by the dash-dot black line, and the lower bound $y(t)$ and the upper bound $\bar{y}(t)$ of the output are depicted by solid red lines. From the initial condition (0.4, 2.5), the outputs using our method and that in Tee et al. (2011) are shown in dash-dot blue and dash-dot magenta lines, respectively; from the initial condition $(-0.3, -2)$, the outputs using our method and that in Tee et al. (2011) are shown in dash blue and dash magenta lines, respectively. It can be seen that both methods satisfy the output constraint for all time and can track y_d after some time. However, the output using our method has a shorter transient time, and it starts tracking y_d without deviating to the constraint boundary (partially because the nominal control is designed independent of the constraints).

Next, we consider a reference trajectory $y_d(t) = \sin t$, which violates the constraints at some time, and initial condition (1.4, 0), which implies that the initial output lies outside the constraint region. Note that this case cannot be solved by the BLF method in Tee et al. (2011). Using our QP-based method, we still compute the nominal tracking control law \hat{u} by the backstepping approach, and obtain u by solving (CBF-QP). The simulation result using our method is shown in Fig. 2. It can be seen that the output y enters the constraint region after some time and stays inside it afterwards; moreover, y tracks y_d when $\underline{y} \leq y_d \leq \bar{y}$, tracks \bar{y} when $y_d > \bar{y}$, and tracks \underline{y} when $y_d < \underline{y}$.

Results above for SISO systems can be generalized to MIMO systems without difficulty. The CBF candidates can be defined

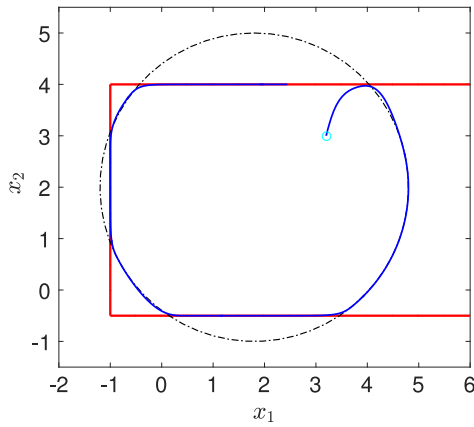


Fig. 3. The output trajectory (solid blue line) from the initial condition (3.2, 3, 0, 0) where the red lines are the constraint boundaries. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

similar to (13) and (14), such that non-negativeness of the CBFs implies satisfaction of the output constraints. Sufficient conditions to ensure the control-sharing property of multiple CBFs can be given easily similar to Theorem 2. Hence the input can be obtained by solving a quadratic program, such that the constraints are always respected and the tracking of a reference trajectory is achieved as much as possible. We use the following example from Jiang and Astolfi (2016) to illustrate the design procedure for MIMO systems.

Example 2. Consider a system described by

$$\begin{aligned}\dot{x}_1 &= x_3, \\ \dot{x}_2 &= x_4, \\ \dot{x}_3 &= -x_1 - 0.5x_2 - x_3 - 0.3x_4 + u_1, \\ \dot{x}_4 &= -0.4x_1 - 2x_2 - 0.3x_3 + 0.5x_4 + u_2.\end{aligned}$$

The outputs of the system are $y_1 = x_1, y_2 = x_2$, the output constraints are $y_1(t) \geq -1, -0.5 \leq y_2(t) \leq 4$, and the reference trajectories for y_1 and y_2 are $y_{1d}(t) = 1.8 + 3 \sin(t + 0.5)$ and $y_{2d}(t) = 2 + 3 \cos(t + 0.5)$, respectively. The outputs of the system have well-defined, global relative degrees $\{2, 2\}$, for all $x \in \mathbb{R}^4$. Note that trajectories of y_{1d} and y_{2d} constitute a circle in the x_1 - x_2 plane with centre (1.8, 2) and radius 3, which intersects with boundaries of the constraint region.

We define CBF candidates $h_1^1(t, x) = x_1 + 1, h_2^1(t, x) = x_2 + 0.5, h_2^2(t, x) = 4 - x_2$, which all have a relative degree 2. Since $\bar{y}_2 - \underline{y}_2 = 4.5$ is constant, it is easy to check that h_1^1, h_2^1, h_2^2 are CSBFs, provided that the corresponding $a_1, a_2 > 0$ are chosen such that all the roots of $\lambda^2 + a_1\lambda + a_2$ are negative. We choose here $a_1 = 20, a_2 = 100$. The nominal tracking control law \hat{u} is obtained via the backstepping approach. Simulation results for two different initial conditions are shown in Figs. 3 and 4, respectively, where the boundary of the constraint region is depicted by solid red lines, the reference trajectory is depicted by the dash-dot black line, and the output trajectory is depicted by the solid blue line.

In Fig. 3, the initial condition is chosen as (3.2, 3, 0, 0) where the initial output is inside the constraint region. It can be seen that the output trajectories are always within the constraint region, and track the reference trajectory well when it is inside the constraint region. Note that the nominal controller \hat{u} here can be replaced by human operators as was considered in Jiang and Astolfi (2016). In Fig. 4, the initial condition is chosen as (2, 4.5, 0, 0) where the initial output is outside the constraint region. It can be seen that

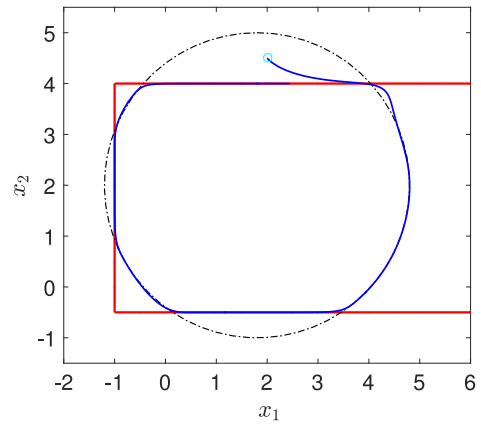


Fig. 4. The output trajectory (solid blue line) from the initial condition (2, 4.5, 0, 0) where the red lines are the constraint boundaries. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the output trajectories enter the constraint region after some time and stay inside it afterwards; the output trajectories also track the reference trajectory well when it is inside the constraint region as expected.

4. Conclusions

In this paper, we studied the control-sharing property of multiple CBFs, and investigated the tracking control problem of input-output linearizable systems with multiple output constraints under a QP-based framework. The QP is guaranteed to be feasible because of the control-sharing property of CBFs. Several advantages of our control design method over existing ones have been discussed and shown by examples. Future work include relaxing the global relative degree assumption, and considering systems with disturbances and input constraints.

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Xiangru Xu received his B.S. degree in applied mathematics from Beijing Normal University in 2009, and his Ph.D. degree in control from Chinese Academy of Sciences in 2014. Since 2014, he is a postdoc research fellow in the Department of Electrical Engineering and Computer Science in the University of Michigan, Ann Arbor, USA.