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Disturbance Observer-based Robust Control Barrier Functions

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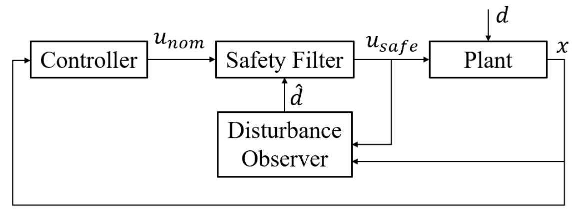
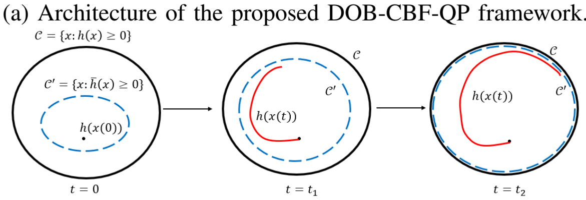
***Abstract*— This work presents a safe control design approach that integrates the disturbance observer (DOB) and the control barrier function (CBF) for systems with external disturbances. Different from existing robust CBF results that consider the "worst case" of disturbances, this work utilizes a DOB to estimate and compensate for the disturbances. DOB-CBF-based controllers are constructed with provably safe guarantees by solving convex quadratic programs online, to achieve a better tradeoff between safety and performance. Two types of systems are considered individually depending on the magnitude of the input and disturbance relative degrees. The effectiveness of the proposed methods is illustrated via numerical simulations.**



(a) Architecture of the proposed DOB-CBF-QP framework.



(b) Illustration of the evolution of the safe set $\mathcal{C}'$.

Fig. 1: As shown in (b), trajectories of the closed-loop system are guaranteed to stay in a set $\mathcal{C}'$. Since DOB will ensure $\hat{d}$ converge to d, the set $\mathcal{C}'$ keeps expanding and can be made arbitrarily close to the original safe set, $\mathcal{C}$, by choosing parameters appropriately. In contrast, robust CBF design based on the "worst-case" of disturbances will result in a safe set whose size shrinks as the magnitude of disturbances becomes larger. Note that safety, instead of input-to-state safety, is ensured by the DOB-CBF-QP-based controller.

I. Introduction

Control barrier functions (CBFs) have emerged as one powerful tool for ensuring control system safety in the form of set invariance and have been successfully applied to various autonomous and robotic systems [1]. Nevertheless, most existing works on CBF-based control design rely on accurate model information and state measurement, which are usually difficult to obtain in practice. To address this problem, robust and adaptive CBF approaches were proposed for systems with model/measurement uncertainties and/or external disturbances [2], [3], [4], [5], [6], [7]. Most of the robust CBF-based methods consider the "worst case" of disturbances and design safe controllers that are often unnecessarily conservative.

Recently, some robust CBF control schemes based on disturbance estimation and compensation techniques were proposed with the goal of reducing the conservatism of the related safe controllers [8], [9], [10], [11], [12]. For example, in [8], [9], a high-gain input disturbance observer was integrated into the CBF framework; in [10], Gaussian processes were employed to estimate the disturbances/uncertainties from data and an end-to-end safe reinforcement learning scheme was developed based on CBFs; in [11], a piecewise-constant disturbance estimation law was proposed and integrated into the robust CBF framework; in [12], a CBF-based safe control law was designed for autonomous surface vehicle systems based on the fixed-time extended state observer.

Disturbance observer (DOB) is a special class of unknown input observers. DOBs estimate the internal and external disturbances by using identified dynamics and measurable states of plants, and have been widely employed in applications such as robotics, automotive, and power electronics [13], [14], [15]. In contrast to other worst-cased-based robust control schemes, the DOB-based methods aim to attenuate the influence of disturbances by compensating for the disturbances and achieve a better tradeoff between robustness and performance. The majority of existing DOB-based control schemes focus on systems whose disturbance relative degree is higher than or equal to the input relative degree [16]; however, systems with a lower disturbance relative degree are ubiquitous (e.g., the missile system [16] and the flexible joint manipulator [17]), and various results were recently proposed to design DOBs for such systems.

This paper develops a safe control design method that integrates the DOB and the CBF for systems with external disturbances. As shown in Fig. 1(a), a DOB is introduced to generate an estimate of the disturbance, which is used by the CBF-based safety filter to compensate for the disturbance. By solving a convex quadratic program (QP) online, a safe control law is obtained that can achieve a better tradeoff between safety and performance. Specifically, this paper first presents a DOB-CBF-QP-based safe control design method for systems whose input relative degree is not higher than the disturbance relative degree. Compared with existing results, the proposed approach relies on milder assumptions and can provide a robust safety guarantee even if the bound of the

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disturbances is not exactly known. This paper also establishes two DOB-CBF-QP-based approaches for a class of systems whose input relative degree is lower than the disturbance relative degree, by using the recursive CBF design and the extended DOB techniques, respectively. To the best of our knowledge, this is the first safe control design result for such class of systems. The remainder of this paper is organized as follows: preliminaries about CBFs and DOBs and the problem statement are provided in Section II, the main results are presented in Section III, numerical simulation results are provided to validate the proposed methods in Section IV, and finally, the conclusion is drawn in Section V.

II. Preliminaries & Problem Statement

A. Control Barrier Function

Consider a system

$$\dot{x} = f(x) + g_1(x)u + g_2(x)d(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ are known and sufficiently smooth functions, and $d(t) : \mathbb{R} \to \mathbb{R}^q$ represents the unknown external disturbance.

Suppose that $h(x) : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth function associated with system (1). If $m = q = 1$, then system (1) is said to have an *input relative degree* of $r(1 \le r \le n)$ in a region $\mathcal{D}$ if $L_{g_1}L_f^{r-1}h(x_0) \neq 0, L_{g_1}L_f^i h(x) = 0, i = 1, \cdots, r-2$, for all $x \in \mathcal{D}$, and an *disturbance relative degree* of $\rho(1 \le \rho \le n)$ if $L_{g_2}L_f^{\rho-1}h(x_0) \neq 0$, $L_{g_2}L_f^j h(x) = 0, j = 1, \cdots, \rho-2$, for all $x \in \mathcal{D}$, where $L_{g_s}L_f^t h$ are Lie derivatives [16], [18], [19]. If $m > 1$ and/or $q > 1$, then the vector input and disturbance relative degrees can be similarly defined following [18, Section 5].

In [20] and [21], (zeroing) CBFs with different relative degrees were introduced for disturbance-free systems. Specifically, given system (1) with $g_2 = 0$ and a safe set $\mathcal{C} \subset \mathbb{R}^n$ defined by

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \ge 0\}, \tag{2}$$

the function h is called a CBF of (input) relative degree 1 if

$$\sup_u \left[L_f h(x) + L_{g_1}h(x)u + \gamma h(x)\right] \ge 0 \tag{3}$$

for all $x \in \mathbb{R}^n$, where $\gamma > 0$ is a given positive constant. It was proven in [20] that if $h(x(0)) > 0$, then any Lipschitz continuous control input $u(x) \in \{u \mid L_f h(x) + L_{g_1}h(x)u + \gamma h(x) \ge 0\}$ will ensure the forward invariance of $\mathcal{C}$. Similarly, the function h is called a CBF of (input) relative degree $r(r \ge 2)$ if there exists $\mathbf{a} \in \mathbb{R}^r$, such that

$$\sup_u [L_{g_1}L_f^{r-1}h(x)u + L_f^r h(x) + \mathbf{a}^\top \eta(x)] \ge 0, \tag{4}$$

for all $x \in \mathbb{R}^n$, where $\eta(x) = [L_f^{r-1}h, L_f^{r-2}h, ..., h]^\top \in \mathbb{R}^r$, and $\mathbf{a} = [a_1, ..., a_r]^\top \in \mathbb{R}^r$ is a set of parameters chosen such that the roots of $\lambda^r + a_1\lambda^{r-1} + ... + a_{r-1}\lambda + a_r = 0$ are all negative reals $-\lambda_1, ..., -\lambda_r < 0$. The functions $s_k(x(t))$ for $k = 0, 1, ..., r$ are defined recursively as

$$s_0(x(t)) = h(x(t)),\ s_k(x(t)) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_k\right) \circ s_{k-1}. \tag{5}$$

If $s_k(x(0)) > 0$ for $k = 0, 1, ..., r-1$, then any Lipschitz continuous control input $u(x) \in \{u \mid L_{g_1}L_f^{r-1}h(x)u + L_f^r h(x) + \mathbf{a}^\top \eta(x) \ge 0\}$ will ensure the forward invariance of $\mathcal{C}$ [21], [22].

B. Disturbance Observer & Extended Disturbance Observer

1) Disturbance Observer: We follow the results of [23] to introduce the DOB that will be used for safe control design in Section III. A standard assumption for DOB is given first.

Assumption 1: The disturbance $d(t)$ and its derivative $\dot{d}(t)$ are bounded by known positive constants, i.e., $\|d(t)\| \le \omega_0$ and $\|\dot{d}(t)\| \le \omega_1, \forall t > 0$ where $\omega_0 > 0$ and $\omega_1 > 0$.

Given system (1), we consider the following DOB:

$$\begin{cases} \hat{d} = z + \alpha p, \\ \dot{z} = -\alpha L_d(f + g_1 u + g_2\hat{d}), \end{cases} \tag{6}$$

where $\hat{d}$ is the disturbance estimation, $L_d(x)$ is the observer gain satisfying $-x^\top L_d g_2 x \le -x^\top x$ for any x (e.g., $L_d = -(g_2^\top g_2)^{-1}g_2^\top$ if $g_2(x)$ has a full column rank), $\alpha > 0$ is a positive *tuning parameter*, and $p(x)$ is a function satisfying $\frac{\partial p}{\partial x} = L_d(x)$. The design of $p(x)$ and L_d is non-trivial and problem-specific; see [13], [14], [15] for more details.

Define the *disturbance estimation error* as

$$e_d = \hat{d} - d. \tag{7}$$

Then, $\dot{e}_d = \dot{\hat{d}} - \dot{d} = \dot{z} + \alpha\frac{\partial p}{\partial x}\dot{x} - \dot{d} = \dot{z} + \alpha L_d\dot{x} - \dot{d}$. Substituting (1) and (6) into this equality yields $\dot{e}_d = -\alpha L_d g_2 e_d - \dot{d}$. Choose a Lyapunov candidate function $V_1 = \frac{1}{2}\|e_d\|^2$. Invoking Assumption 1 and the definition of L_d, we have

$$\dot{V}_1 \le -2\kappa V_1 + \frac{\omega_1^2}{2\nu_1}, \tag{8}$$

where $\kappa \triangleq \alpha - \frac{\nu_1}{2}$, ν_1 is a constant satisfying $0 < \nu_1 < 2\alpha$, and the second inequality is from the fact that $\omega_1\|e_d\| \le \frac{\nu_1}{2}\|e_d\|^2 + \frac{1}{2\nu_1}\omega_1^2$. Recalling comparison lemma [19], we have

$$\|e_d(t)\| \le \sqrt{\frac{2\nu_1\kappa\|e_d(0)\|^2 e^{-2\kappa t} + \omega_1^2(1 - e^{-2\kappa t})}{2\nu_1\kappa}}. \tag{9}$$

Form (9) one can see the disturbance estimation error e_d is uniformly ultimately bounded.

2) Extended Disturbance Observer: As a generalization of DOB, the extended DOB was proposed in [17] to estimate the high order derivatives of disturbances. Consider the following system

$$\dot{x} = f(x, u) + d(t) \tag{10}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $d : \mathbb{R} \to \mathbb{R}^n$ is the external disturbance. Similar to Assumption 1, we assume d and its derivatives are bounded.

Assumption 2: The disturbance d and its derivatives $d^{(1)}, ..., d^{(r)}$, where r is a fixed positive integer, are bounded by some known constants, i.e., $\|d_E(t)\| \le \xi_0$ and $\|d^{(r+1)}(t)\| \le \xi_1$ for any $t > 0$, where $\xi_0 > 0, \xi_1 > 0$, and $d_E(t) \triangleq \begin{bmatrix} d^{(0)\top} & d^{(1)\top} & \cdots & d^{(r)\top} \end{bmatrix}^\top$ with $d^{(0)} = d$.

Consider the following extended DOB as in [17]:

$$\hat{d}^{(i)} = p_i + l_i x, \tag{11a}$$

$$\dot{p}_i = -l_i(f + \hat{d}^{(0)}) + \hat{d}^{(i+1)}, \tag{11b}$$

where $\hat{d}^{(i)}$ denotes the estimate of $d^{(i)}$, l_i is a tuning parameter, $i = 0, 1, \cdots, p$, and $\hat{d}^{(r+1)} = 0$. Define $\hat{d}_E(t) =$

$\begin{bmatrix}\hat{d}^{(0)\top} & \hat{d}^{(1)\top} & \cdots & \hat{d}^{(r)\top}\end{bmatrix}^\top$ and the estimation error

$$\tilde{e}_d = \hat{d}_E - d_E. \tag{12}$$

Equations (11) can be written compactly as

$$\dot{\tilde{e}}_d = A\tilde{e}_d + Bd^{(r+1)}, \tag{13}$$

where

$$A = \begin{bmatrix} -l_0 & 1 & 0 & \cdots & 0 \\ -l_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -l_{r-1} & 0 & 0 & \cdots & 1 \\ -l_r & 0 & 0 & \cdots & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix},$$

and $l_0, \cdots, l_p$ are selected such that $\lambda_{\min}(A) < 0$. Define a Lyapunov candidate function as $V_2 = \frac{1}{2}\|\tilde{e}_d\|^2$. Then,

$$\dot{V}_2 \le -2\kappa_E V_2 + \frac{\xi_1^2}{2\nu_2}, \tag{14}$$

where $\kappa_E \triangleq -\lambda_{\min}(A) - \frac{\nu_2}{2}$ and ν_2 is a constant satisfying $0 < \nu_2 < -2\lambda_{\min}(A)$. Similar to (9),

$$\|\tilde{e}_d(t)\| \le \sqrt{\frac{2\nu_2\kappa_E\|\tilde{e}_d(0)\|^2 e^{-2\kappa_E t} + \xi_1^2(1-e^{-2\kappa_E t})}{2\nu_2\kappa_E}}. \tag{15}$$

It can be seen the $\tilde{e}_d$ is uniformly ultimately bounded.

C. Problem Statement

In this paper, we will consider the DOB-CBF-based safe control design problem for two types of systems individually depending on the magnitude of the input and disturbance relative degrees. In the first problem, the system has an input relative degree not higher than its disturbance relative degree.

Problem 1: Given system (1) whose input relative degree is not higher than its disturbance relative degree, the safe set $\mathcal{C}$ defined in (2), and the DOB given in (6), design a controller $u(x, \hat{d})$ such that the closed-loop system is safe with respect to $\mathcal{C}$, i.e., $h(x(t)) \ge 0$ for all $t \ge 0$ provided that $x(0) \in \mathcal{C}$.

For the second problem, we consider the following system with a mismatched disturbance:

$$\begin{aligned} \dot{x}_1 &= f_1(\bar{x}_2) + d, \\ \dot{x}_2 &= f_2(\bar{x}_3), \\ &\cdots \\ \dot{x}_n &= f_n(\bar{x}_n) + g(\bar{x}_n)u, \end{aligned} \tag{16}$$

where $x_i \in \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ is the mismatched disturbance, and $\bar{x}_i = [x_1\ x_2\ \cdots\ x_i]^\top \in \mathbb{R}^i$, $i = 1, 2, \cdots, n$. The safe set for such a system is given as

$$\tilde{\mathcal{C}} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : h(x_1) \ge 0\}, \tag{17}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a C^n function. Clearly, for system (16) with the output function h defined in (17), its disturbance relative degree is lower than its input relative degree.

Problem 2: Given system (16) and the safe set $\tilde{\mathcal{C}}$ defined in (17), design a DOB with the estimated disturbance $\hat{d}$ and a controller $u(x, \hat{d})$ such that the closed-loop system is safe with respect to $\tilde{\mathcal{C}}$, i.e., $h(x_1(t)) \ge 0$ for all $t \ge 0$ provided that $\bar{x}_n(0) \in \tilde{\mathcal{C}}$.

Remark 1: In this paper we only consider the single-input-single-output system (16) with one disturbance due to the page limit. However, the proposed methods can be readily extended to more general systems, such as the nonlinear missile model studied in Example 2. The detailed design procedure will be given in our future work.

III. Main Results

In this section, the main results of this paper are presented. In Section III-A, a DOB-CBF-based QP is proposed for the system whose input relative degree is not higher than its disturbance relative degree. In Section III-B, two DOB-CBF-based QPs are developed for the system whose input relative degree is higher than its disturbance relative degree, by using recursive CBF design and extended DOB techniques, respectively; compared with the first approach, the second one tends to have less conservative safe controller in simulation but it requires more restrictive assumptions (see simulation examples in Section IV).

A. DOB-CBF-QP for Solving Problem 1

In this subsection, we will present the DOB-CBF-based safe control design method to solve Problem 1. We will first consider the simple case where the CBF h has an input relative degree 1, and then generalize the result to the case where h has a higher input relative degree $r(r > 1)$.

The following result is the first main result of this work for the CBF h with an input relative degree 1.

Theorem 1: Consider the system (1), the safe set $\mathcal{C}$ defined in (2), and the DOB given in (6) with $\hat{d}(0) = 0$. Suppose that Assumption 1 holds, h has an input relative degree 1, and $h(x(0)) > 0$. Assume that there exist positive constants $\gamma, \alpha, \beta > 0$ such that $\alpha > \frac{\gamma+\nu_1}{2}$, $\beta > \frac{\|e_d(0)\|^2}{2h(x(0))}$, and $\sup_u[L_f h - \|L_{g_2}h\|\chi - \frac{\omega_1^2}{2\nu_1\beta} - \frac{\beta\|L_{g_2}h\|^2}{4\alpha-2\nu_-2\gamma} + \gamma h + L_{g_1}hu] \ge 0$ hold true, where $\chi = \omega_0 + \sqrt{\omega_0^2 + \frac{\omega_1^2}{2\nu_1\kappa}}$. Then, any Lipschitz continuous controller, $u(x) \in K_{DOB}(x, \hat{d}) \triangleq \{u \mid \psi_0 + \psi_1 u \ge 0\}$ where

$$\begin{aligned} \psi_0 &= L_f h + L_{g_2}h\hat{d} - \frac{\omega_1^2}{2\nu_1\beta} - \frac{\beta\|L_{g_2}h\|^2}{4\alpha-2\nu_1-2\gamma} + \gamma h, \\ \psi_1 &= L_{g_1}h, \end{aligned}$$

will guarantee $h(x(t)) \ge 0$ for all $t \ge 0$.

Proof: If $\hat{d}(0) = 0$, then $\hat{d}(t)$ satisfies $\|\hat{d}(t)\| = \|d(t) + e_d(t)\| \le \chi$ from (9). Define a new candidate CBF $\bar{h}$ as $\bar{h}(x(t), t) = \beta h - \frac{1}{2}e_d^\top e_d$ where e_d is defined in (7). It can be seen $\bar{h} \ge 0$ implies $h \ge \frac{\|e_d\|^2}{2\beta} \ge 0$. Since $\beta > \frac{\|e_d(0)\|^2}{2h(x(0))}$, one can verify $\bar{h}(x(0), 0) > 0$. Moreover, $\dot{\bar{h}}$ satisfies

$$\begin{aligned} \dot{\bar{h}} &\overset{(8)}{\ge} \beta(L_f h + L_{g_1}hu + L_{g_2}hd) + \kappa e_d^\top e_d - \frac{\omega_1^2}{2\nu_1} \\ &= \beta(L_f h + L_{g_1}hu + L_{g_2}h\hat{d} - L_{g_2}he_d) + \frac{\gamma}{2}e_d^\top e_d \\ &\quad + \left(\alpha - \frac{\nu_1}{2} - \frac{\gamma}{2}\right)e_d^\top e_d - \frac{\omega_1^2}{2\nu_1} \\ &= \beta(L_f h + L_{g_1}hu + L_{g_2}h\hat{d}) - \frac{\omega_1^2}{2\nu_1} - \frac{\beta^2\|L_{g_2}h\|^2}{4\alpha - 2\nu_1 - 2\gamma} \\ &\quad + \left\|\sqrt{\alpha - \frac{\nu_1}{2} - \frac{\gamma}{2}}e_d^\top - \frac{\beta L_{g_2}h}{2\sqrt{\alpha - \frac{\nu_1}{2} - \frac{\gamma}{2}}}\right\|_2^2 + \frac{\gamma}{2}e_d^\top e_d \end{aligned}$$

$$\geq \beta\left(\psi_0+\psi_1 u-\gamma h\right)+\frac{\gamma}{2} e_d^{\top} e_d.$$

Therefore, any $u \in K_{DOB}$ yields $\dot{\bar{h}} \geq -\gamma\beta h+\frac{\gamma}{2} e_d^{\top} e_d \geq -\gamma\left(\beta h-\frac{1}{2} e_d^{\top} e_d\right)=-\gamma\bar{h}$, which implies that $\bar{h}(x(t), t) \geq 0, \forall t \geq 0$ as $\bar{h}(x(0), 0) \geq 0$. Thus, $h(x(t)) \geq 0, \forall t>0$. ■

The safe controller proposed in Theorem 1 is obtained by solving the following DOB-CBF-QP:

$$\begin{aligned} \min_u \quad & \|u-u_{nom}\|^2 \\ \text{s.t.} \quad & \psi_0+\psi_1 u \geq 0, \\ & \text{DOB given in (6),} \end{aligned} \tag{18}$$

where ψ_0, ψ_1 are given in Theorem 1 and u_{nom} is any given nominal control law.

Remark 2: The proof of Theorem 1 reveals that, by ensuring $\bar{h}(t) \geq 0$, $x(t)$ is restricted to stay in a set defined by $\mathcal{C}'(t) \triangleq \{x \mid h(x) \geq \frac{\|e_d(t)\|^2}{2\beta}\}$. According to (9), the ultimate bound of $\|e_d(t)\|$ is $\frac{\omega_1}{\sqrt{\nu_1(2\alpha-\nu_1)}}$; therefore, $\mathcal{C}'(t)$ will eventually converge to the set $\mathcal{C}'(\infty) \triangleq \{x \mid h(x) \geq \frac{\omega_1^2}{2\beta\nu_1(2\alpha-\nu_1)}\}$. By choosing the parameters α, ν_1, β appropriately (e.g., choose α or β large enough with other parameters fixed), the set $\mathcal{C}'(\infty)$ can be made arbitrarily close to the original safe set $\mathcal{C}$ despite the unknown disturbance; that is, the system trajectory is allowed to approach arbitrarily close to the boundary of $\mathcal{C}$ (see also Fig. 1 (b)). Note that the selection of β depends on the estimation of $e_d(0)$. In practice, β can be selected sufficiently large such that $\beta>\frac{\|e_d(0)\|^2}{2h(x(0))}$.

Remark 3: The assumptions in Theorem 1 are milder than those in [8]. If $K_{DOB}(x, \hat{d})$ is modified by dropping the term $\frac{\omega_1^2}{2\nu_1\beta}$ in ψ_0, then any $u(x) \in K_{DOB}(x, \hat{d})$ yields $\dot{\bar{h}} \geq -\gamma\bar{h}-\frac{\omega_1^2}{2\nu_1}$, which implies input-to-state safety of the system [24], [25]. Therefore, even when ω_1 is not exactly known, the proposed controller may be modified to serve as an input-to-state safe controller; in contrast, the exact value of ω_1 is indispensable in the control design of the DOB-CBF-based method in [8].

Now we consider the case where a CBF h has a higher input relative degree $r (r>1)$. To simplify the expression, we assume u and d in (1) are scalars; however, the proposed method can be readily extended to the case where u and d are vectors. In Section IV, we will present a robot manipulator example whose input and disturbance are both vectors.

Theorem 2: Consider the system (1) with dimensions $m=q=1$, the safe set $\mathcal{C}$ defined in (2), and the DOB given in (6) with $\hat{d}(0)=0$. Suppose that Assumption 1 holds, h has an input relative degree $r (r>1)$, and $s_k(x(0))>0$ for $k=0,1,...,r-1$, where $s_k(x(t))$ is defined in (5). Assume that there exist $\mathbf{a} \in \mathbb{R}^r$, $\beta>0$, $\alpha>\frac{\lambda_r+\nu_1}{2}$, such that $\sup_u[L_f^r h-\|L_{g_2} L_f^{r-1} h\|\chi-\frac{\omega_1^2}{2\nu_1\beta}-\frac{\beta\|L_{g_2} L_f^{r-1} h\|^2}{4\alpha-2\lambda_r-2\nu}+\mathbf{a}^{\top}\eta(x)+L_{g_1} L_f^{r-1} h u] \geq 0$ and $\beta>\frac{\|e_d(0)\|^2}{2s_{r-1}(x(0))}$ hold true, where $\mathbf{a}$, η are defined in (4) and $\chi=\omega_0+\sqrt{\omega_0^2+\frac{\omega_1^2}{2\nu_1\kappa}}$. Then any Lipschitz continuous controller $u(x) \in K_{DOB}^r(x, \hat{d}) \triangleq \{u \mid \psi_0^r+\psi_1^r u \geq 0\}$, where

$$\psi_0^r=L_f^r h+L_{g_2} L_f^{r-1} h\hat{d}-\frac{\omega_1^2}{2\nu_1\beta}-\frac{\beta\|L_{g_2} L_f^{r-1} h\|^2}{4\alpha-2\lambda_r-2\nu}+\mathbf{a}^{\top}\eta(x),$$
$$\psi_1^r=L_{g_1} L_f^{r-1} h,$$

will guarantee $h(x(t)) \geq 0$ for all $t \geq 0$.

Proof: We define a new CBF candidate as $\bar{h}^r(x, t)=\beta s_{r-1}(x)-\frac{1}{2} e_d^{\top} e_d$. It can be easily verified that selecting $u \in K_{DOB}^r$ gives $\dot{\bar{h}}^r \geq -\lambda_r\bar{h}^r$, and $\beta>\frac{\|e_d(0)\|^2}{2s_{r-1}(x(0))}$ indicates $\bar{h}^r(x(0)) \geq 0$. Therefore, one can see that $\bar{h}^r \geq 0$, which indicates $h(x_1(t)) \geq 0$ for any $t>0$ because $s_k(x(0))>0$, $k=0,1,\cdots,r-1$ [21]. ■

The safe controller proposed in Theorem 2 is obtained by solving the following DOB-CBF-QP:

$$\begin{aligned} \min_u \quad & \|u-u_{nom}\|^2 \\ \text{s.t.} \quad & \psi_0^r+\psi_1^r u \geq 0, \\ & \text{DOB given in (6),} \end{aligned} \tag{19}$$

where ψ_0^r, ψ_1^r are given in Theorem 2 and u_{nom} is any given nominal control law.

B. DOB-CBF-QP for Solving Problem 2

In this subsection, we will present two DOB-CBF-based safe control design methods to solve Problem 2. The first method relies on a DOB and recursive CBF design, while the second method is based on an extended DOB. We consider the system (16) and the safe set defined in (17).

1) DOB and Recursive CBF-Based Method: In this method, we design the following DOB as given in (6):

$$\begin{aligned} \hat{d} &= z+\alpha x_1, \\ \dot{z} &= -\alpha(f_1(x_1, x_2)+\hat{d}), \end{aligned} \tag{20}$$

From (20) we have $\dot{\hat{d}}=-\alpha e_d$, where e_d is defined in (7). Define a set of functions $\bar{h}_i(\bar{x}_{i+1}, t)$, $i=0,1,\cdots,n-1$ as follows:

$$\bar{h}_0(x_1, t)=h(x_1), \tag{21}$$

$$\bar{h}_i(\bar{x}_{i+1}, t)=\mathcal{Q}_i(\bar{x}_{i+1}, \hat{d})-\beta_i V_1, \tag{22}$$

where $V_1=\frac{1}{2} e_d^2$, $\mathcal{Q}_i(\bar{x}_{i+1}, \hat{d}), i=1,2,\cdots,n-1$, is defined recursively as

$$\mathcal{Q}_1(\bar{x}_2, \hat{d})=\frac{\partial h}{\partial x_1}(f_1+\hat{d})-\frac{1}{2\beta_1}\left(\frac{\partial h}{\partial x_1}\right)^2+\lambda_1 h, \tag{23a}$$

$$\mathcal{Q}_i(\bar{x}_{i+1}, \hat{d})=\frac{\partial \mathcal{Q}_{i-1}}{\partial x_1}(f_1+\hat{d})+\sum_{j=2}^{i} \frac{\partial \mathcal{Q}_{i-1}}{\partial x_j} f_j+\lambda_i \mathcal{Q}_{i-1} -\frac{\left(\alpha\frac{\partial \mathcal{Q}_{i-1}}{\partial \hat{d}}+\frac{\partial \mathcal{Q}_{i-1}}{\partial x_1}\right)^2}{\beta_{i-1}\left(4\alpha-2\nu_1-2\lambda_i\right)}-\frac{\beta_{i-1}\omega_1^2}{2\nu_1}, \tag{23b}$$

and $0<\lambda_i<2\alpha-\nu_1$, $\beta_i>0$ are tuning parameters. Define

$$\mathcal{P}(\bar{x}_n, \hat{d})=\frac{\partial \mathcal{Q}_{n-1}}{\partial x_1}(f_1+\hat{d})+\sum_{j=2}^{n-1} \frac{\partial \mathcal{Q}_{n-1}}{\partial x_j} f_j-\frac{\beta_{n-1}\omega_1^2}{2\nu_1} -\frac{\left(\alpha\frac{\partial \mathcal{Q}_{n-1}}{\partial \hat{d}}+\frac{\partial \mathcal{Q}_{n-1}}{\partial x_1}\right)^2}{\beta_{n-1}\left(4\alpha-2\nu_1-2\gamma\right)}.$$

It is easy to verify that $\frac{\partial \mathcal{Q}_{n-1}}{\partial x_n}$ is independent of $\hat{d}$. Meanwhile, if $x_1, \cdots, x_n$ are fixed, $\mathcal{P}(\bar{x}_n, \hat{d})$ and $\mathcal{Q}_{n-1}(\bar{x}_n, \hat{d})$ can be represented as polynomials of $\hat{d}$. Hence, one can see that there exist functions $\mathcal{F}_i(\bar{x}_n), i = 0, 1, \cdots, n-1$ and $\mathcal{G}_j(\bar{x}_n), j = 0, 1, \cdots, n$ such that $\mathcal{Q}_{n-1}(\bar{x}_n, \hat{d})$ and $\mathcal{P}(\bar{x}_n, \hat{d})$ can be expressed as

$$\mathcal{Q}_{n-1}(\bar{x}_n, \hat{d}) = \mathcal{F}_0(\bar{x}_n) + \sum_{i=1}^{n-1} \mathcal{F}_i(\bar{x}_n)\hat{d}^i, \quad (24a)$$

$$\mathcal{P}(\bar{x}_n, \hat{d}) = \mathcal{G}_0(\bar{x}_n) + \sum_{i=1}^{n} \mathcal{G}_i(\bar{x}_n)\hat{d}^i. \quad (24b)$$

Theorem 3: Consider the system (16), the safe set $\tilde{\mathcal{C}}$ defined in (17), and the DOB given in (20) with $\hat{d}(0) = 0$. Suppose that Assumption 1 holds, $\bar{h}_i(\bar{x}_{i+1}(0), 0) \geq 0$, and $i = 0, 1, \cdots, n-1$, where $\bar{h}_i$ is defined in (22). Assume that there exist $\gamma > 0, \lambda_i > 0, \beta_i > 0$ for $i = 1, \cdots, n-1$, such that $\gamma < 2\alpha - \nu_1$ and $\sup_u [\mathcal{G}_0 + \gamma\mathcal{F}_0 - (|\mathcal{G}_n|\chi^n + \sum_{i=1}^{n-1} |\mathcal{G}_i + \gamma\mathcal{F}_i|\chi^i) + \frac{\partial \mathcal{Q}_{n-1}}{\partial x_n} u] \geq 0$, where $\chi = \omega_0 + \sqrt{\omega_0^2 + \frac{\omega_1^2}{2\nu_1\kappa}}$ and $\mathcal{F}_i$, $\mathcal{G}_i$ are defined in (24). Then any Lipschitz continuous controller $u(x) \in K_{DOB}^{re}(x, \hat{d}) \triangleq \{u \mid \psi_0^{re} + \psi_1^{re} u \geq 0\}$, where

$$\psi_0^{re} = \mathcal{P}(\bar{x}_n, \hat{d}) + \gamma\mathcal{Q}_{n-1}(\bar{x}_n, \hat{d}),$$
$$\psi_1^{re} = \frac{\partial \mathcal{Q}_{n-1}}{\partial x_n},$$

will guarantee $h(x_1(t)) \geq 0$ for all $t \geq 0$.

Proof: We will show that $\bar{h}_i \geq 0$ indicates $\bar{h}_{i-1} \geq 0$ for any $t > 0$ if $\bar{h}_{i-1}(\bar{x}_i(0), 0) \geq 0$, $i = 1, 2, \cdots, n-1$.

Step 1: Note that $\dot{h} + \lambda_1 h$ satisfies $\dot{h} + \lambda_1 h = \frac{\partial h}{\partial x_1}(f_1 + \hat{d}) - \frac{\partial h}{\partial x_1} e_d + \lambda_1 h \geq \frac{\partial h}{\partial x_1}(f_1 + \hat{d}) - \frac{1}{2\beta_1}\left(\frac{\partial h}{\partial x_1}\right)^2 + \lambda_1 h - \frac{\beta_1}{2} e_d^2 = \bar{h}_1(\bar{x}_2, t)$, from which it can be seen that $\bar{h}_1 \geq 0$ indicates $\dot{h} + \lambda_1 h \geq 0$; thus, $h \geq 0$ for any $t > 0$ as $h(x_1(0)) \geq 0$.

Step i ($i = 2, \cdots, n-1$): It can be seen that

$$\begin{aligned}
&\dot{\bar{h}}_{i-1} + \lambda_i \bar{h}_{i-1} \\
\overset{(8)}{\geq} &\frac{\partial \mathcal{Q}_{i-1}}{\partial x_1}(f_1 + \hat{d}) + \sum_{j=2}^{i} \frac{\partial \mathcal{Q}_{i-1}}{\partial x_j} f_j + \lambda_i \mathcal{Q}_{i-1} - \frac{\beta_{i-1}\omega_1^2}{2\nu_1} \\
&- \left(\alpha \frac{\partial \mathcal{Q}_{i-1}}{\partial \hat{d}} + \frac{\partial \mathcal{Q}_{i-1}}{\partial x_1}\right) e_d + \beta_{i-1}\left(\alpha - \frac{\nu_1}{2} - \frac{\lambda_i}{2}\right) e_d^2 \\
\geq &\frac{\partial \mathcal{Q}_{i-1}}{\partial x_1}(f_1 + \hat{d}) + \sum_{j=2}^{i} \frac{\partial \mathcal{Q}_{i-1}}{\partial x_j} f_j + \lambda_i \mathcal{Q}_{i-1} - \frac{\beta_{i-1}\omega_1^2}{2\nu_1} \\
&+ \left[\frac{\left(\alpha \frac{\partial \mathcal{Q}_{i-1}}{\partial \hat{d}} + \frac{\partial \mathcal{Q}_{i-1}}{\partial x_1}\right)}{2\sqrt{\beta_{i-1}\left(\alpha - \frac{\nu_1}{2} - \frac{\lambda_i}{2}\right)}} - \sqrt{\beta_{i-1}\left(\alpha - \frac{\nu_1}{2} - \frac{\lambda_i}{2}\right)} e_d\right]^2 \\
&- \frac{\left(\alpha \frac{\partial \mathcal{Q}_{i-1}}{\partial \hat{d}} + \frac{\partial \mathcal{Q}_{i-1}}{\partial x_1}\right)^2}{\beta_{i-1}(4\alpha - 2\nu_1 - 2\lambda_i)} \geq \mathcal{Q}_i(\bar{x}_{i+1}, \hat{d}),
\end{aligned}$$

from which it can be seen that $\bar{h}_i \geq 0$ indicates $\mathcal{Q}_i \geq 0$ and $\dot{\bar{h}}_{i-1} + \lambda_i \bar{h}_{i-1} \geq 0$; thus, $\bar{h}_{i-1} \geq 0$ for any $t > 0$ because $\bar{h}_{i-1}(x_i(0), 0) \geq 0$.

Step n: Similar to the steps above, one can see that $\dot{\bar{h}}_{n-1} \geq \mathcal{P}(\bar{x}_n, \hat{d}) + \frac{\partial \mathcal{Q}_{n-1}}{\partial x_n} u + \frac{\beta_{n-1}\gamma}{2} e_d^2$. Selecting $u \in K_{DOB}^{re}$ yields $\dot{\bar{h}}_{n-1} \geq -\gamma \bar{h}_{n-1}$, which implies $\bar{h}_{n-1} \geq 0, \forall t > 0$ as $\bar{h}_{n-1}(\bar{x}_n(0), 0) \geq 0$. Thus, $h(x_1(t)) \geq 0$ since $\bar{h}_i(x_{i+1}(0), 0) \geq 0$, $i = 0, 1, \cdots, n-1$. ■

The safe controller proposed in Theorem 3 is obtained by solving the following DOB-CBF-QP:

$$\begin{aligned}
\min_u \quad & \|u - u_{nom}\|^2 \quad (25)\\
\text{s.t.} \quad & \psi_0^{re} + \psi_1^{re} u \geq 0, \\
& \text{DOB given in (20)},
\end{aligned}$$

where ψ_0^{re}, ψ_1^{re} are given in Theorem 3 and u_{nom} is any given nominal control law.

2) Extended DOB-based Method: Now we will present an alternative approach to solve Problem 2 based on the extended DOB. Compared with the DOB-CBF-QP controllers by Theorem 3, controllers obtained from this second approach are often less conservative in simulations (see Section IV).

Consider the system (16) and the safe set defined in (17). Similar to (5), a set of functions $w_k(\bar{x}_{k+1}, t), k = 1, \cdots, n-1$, are defined as

$$w_k(\bar{x}_{k+1}, t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_k\right) \circ w_{k-1}(\bar{x}_k, t), \quad (26)$$

where $w_0(x_1, t) = h(x_1)$ and $\lambda_k > 0$. Suppose that Assumption 2 holds with $r = n-1$, i.e., $\|d_E(t)\| \leq \xi_0$ and $|d^{(n)}| \leq \xi_1$, where $d_E(t)$ is defined in Assumption 2. We design the following extended DOB as in (11):

$$\hat{d}^{(i)} = p_i + l_i x_1, \quad (27a)$$
$$\dot{p}_i = -l_i(f_1 + \hat{d}^{(0)}) + \hat{d}^{(i+1)}, \quad (27b)$$

where $\hat{d}^{(n)}(t) = 0$ for any $t \geq 0$ and $\hat{d}_j^{(i)}$ denotes the estimate of $d_j^{(i)}$, $i = 0, 1, \cdots, n-1$. Suppose that there exist functions $\mathcal{T}_0(\bar{x}_n)$, $\mathcal{T}_1(\bar{x}_n)$, $\mathcal{R}_0(\bar{x}_n)$, $\mathcal{R}_1(\bar{x}_n)$, $\mathcal{R}_2(\bar{x}_n)$, such that

$$w_{n-1} = \mathcal{T}_0(\bar{x}_n) + \mathcal{T}_1(\bar{x}_n) d_E(t), \quad (28a)$$
$$\dot{w}_{n-1} = \mathcal{R}_0(\bar{x}_n) + \mathcal{R}_1(\bar{x}_n) u + \mathcal{R}_2(\bar{x}_n) d_E(t), \quad (28b)$$

where w_{n-1} is defined in (26). Intuitively, condition (28) indicates that w_{n-1} and $\dot{w}_{n-1}$ can be represented as the linear combination of d and its derivatives with fixed $\bar{x}_n$.

Based on the extended DOB given in (27) and the condition shown in (28), the following result provides a QP-based safe controller for solving Problem 2.

Theorem 4: Consider the system (16), the safe set $\tilde{\mathcal{C}}$ defined in (17), and the extended DOB given in (27) with $\hat{d}_E(0) = 0$. Suppose that Assumption 2 holds with $r = n-1$, $w_k(\bar{x}_{k+1}(0), 0) > 0$, $k = 0, 1, \cdots, n-1$, and condition (28) holds. Suppose that there exist positive constants $\gamma > 0$, $\beta > 0$, such that $0 < \gamma < 2\kappa_E$, $\beta > \frac{\|E_d(0)\|^2}{2w_{n-1}(\bar{x}_n(0), 0)}$ and $\sup_u [\mathcal{R}_0 + \gamma\mathcal{T}_0 - \|\mathcal{R}_2 + \gamma\mathcal{T}_1\|\chi_E - \frac{\beta\xi_1^2}{2\nu_2} + \mathcal{R}_1 u - \frac{(\mathcal{R}_2 + \gamma\mathcal{T}_1)^2}{\beta(4\kappa_E - 2\gamma)}] \geq 0$ holds true, where κ_E is defined in (14) and $\chi_E = \xi_0 + \sqrt{\xi_0^2 + \frac{\xi_1^2}{2\nu_2\kappa_E}}$. Then any Lipschitz continuous controller $u(x) \in K_{EDOB}(x, \hat{d}_j^{(i)}) \triangleq \{u \mid \psi_0^E + \psi_1^E u \geq 0\}$ will guarantee $h(x_1(t)) \geq 0$ for all $t \geq 0$, where

$$\psi_0^E = \mathcal{R}_0 + \gamma\mathcal{T}_0 + (\mathcal{R}_2 + \gamma\mathcal{T}_1)\hat{d}_E - \frac{(\mathcal{R}_2 + \gamma\mathcal{T}_1)^2}{\beta(4\kappa_E - 2\gamma)} - \frac{\beta\xi_1^2}{2\nu_2},$$
$$\psi_1^E = \mathcal{R}_1.$$

Proof: If $\hat{d}_E(0)=0$, then $\hat{d}_E(t)$ satisfies $\|\hat{d}_E(t)\| = \|d_E(t)+\tilde{e}_d(t)\| \leq \chi_E$ from (15). Define a new CBF candidate $\bar{h}_E$ as $\bar{h}_E = w_{n-1} - \beta V_2$, where $V_2 = \frac{1}{2}\|\tilde{d}_E\|^2$. Since

$$\begin{aligned}
\dot{\bar{h}}_E &\overset{(14)}{\geq} \mathcal{R}_0 + \mathcal{R}_1 u + \mathcal{R}_2 d_E + \beta\kappa_E\|\tilde{e}_d\|^2 - \frac{\beta\xi_1^2}{2\nu_2} \\
&= \mathcal{R}_0 + \gamma\mathcal{T}_0 + \mathcal{R}_1 u + (\mathcal{R}_2+\gamma\mathcal{T}_1)\hat{d}_E - \gamma(\mathcal{T}_0 + \mathcal{T}_1 d_E) \\
&\quad -(\mathcal{R}_2+\gamma\mathcal{T}_1)\tilde{e}_d + \beta\kappa_E\|\tilde{e}_d\|^2 - \frac{\beta\xi_1^2}{2\nu_2} \\
&\overset{(28)}{\geq} \mathcal{R}_0 + \gamma\mathcal{T}_0 + \mathcal{R}_1 u + (\mathcal{R}_2+\gamma\mathcal{T}_1)\hat{d}_E - \gamma w_{n-1} \\
&\quad + \left\| \sqrt{\beta\left(\kappa_E - \frac{\gamma}{2}\right)} E_d - \frac{(\mathcal{R}_2+\gamma\mathcal{T}_1)}{2\sqrt{\beta\left(\kappa_E-\frac{\gamma}{2}\right)}} \right\|^2 \\
&\quad - \frac{(\mathcal{R}_2+\gamma\mathcal{T}_1)^2}{\beta(4\kappa_E - 2\gamma)} + \frac{\gamma\beta}{2}\|\tilde{e}_d\|^2 - \frac{\beta\xi_1^2}{2\nu_2}. \qquad (29)
\end{aligned}$$

Selecting $u \in K_{EDOB}$ yields $\dot{\bar{h}}_E \geq -\gamma w_{n-1} + \frac{\gamma\beta}{2}\|\tilde{e}_d\|^2 = -\gamma\bar{h}_E$. Note that the selection of β ensures $\bar{h}_E(\bar{x}_n(0),0) \geq 0$. Thus, one can see $\bar{h}_E \geq 0$ and $w_{n-1} \geq 0$ for any $t>0$. As $w_k(\bar{x}_{k+1}(0),0) \geq 0$, $k=0,1,\cdots,n-1$, it easy to see that $h(x_1(t)) \geq 0$ for any $t>0$. ■

The safe controller proposed in Theorem 4 is obtained by solving the following DOB-CBF-QP:

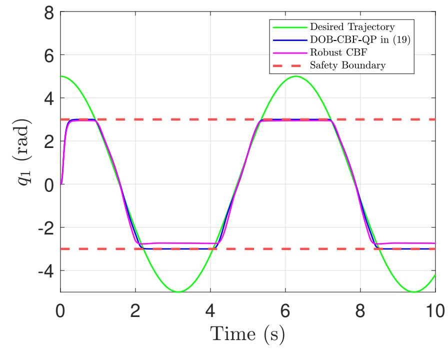
$$\begin{aligned}
\min_{u} \quad & \|u - u_{nom}\|^2 \qquad (30) \\
\text{s.t.} \quad & \psi_0^E + \psi_1^E u \geq 0, \\
& \text{extended DOB given in (27)},
\end{aligned}$$

where ψ_0^E, ψ_1^E are given in Theorem 4 and u_{nom} is any given nominal control law.

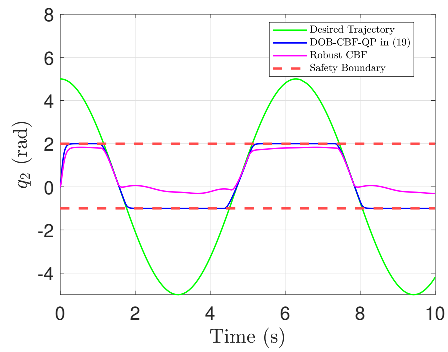
IV. Simulation Examples

In this section, two examples are presented to illustrate the effectiveness of the proposed methods. The robust CBF method proposed in [3] is used for comparison.

Example 1: Consider a 2-DOF planar robot whose dynamics are described by $M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=\tau+\tau_d$, where $q=[q_1\ q_2]^\top \in \mathbb{R}^2$ denote the joint angles, $\tau \in \mathbb{R}^2$ is the control input, and $\tau_d \in \mathbb{R}^2$ represents the external disturbance satisfying $\|\tau_d\| \leq 30$ and $\|\dot{\tau}_d\| \leq 50$. For the robust CBF approach, the bound of τ_d is also set as 30. The physical parameters used are chosen as those in [26], and the nominal controller is a PD controller. The following four CBFs are employed to represent the constraints on joint angles: $h_1 = q_1+3$, $h_2 = 3-q_1$, $h_3 = q_2+1$, and $h_4 = 2-q_2$. Clearly, the input relative degree of the system is equal to the disturbance relative degree. It can be verified that the conditions of Theorem 1 hold, so that a DOB-CBF-QP-based controller can be obtained by solving (18) to ensure the safety of the closed-loop system. The simulation results are presented in Fig. 2. It can be seen that the trajectories of the closed-loop system with the proposed DOB-CBF-QP-based controller by solving (18) are always safe as q_1, q_2 stay inside the respective safe regions bounded by the dashed red lines; furthermore, the trajectories of the closed-loop system are less conservative than the robust CBF approach because the trajectories are able to track the desired trajectories (i.e., the green lines) much better inside the safe region.



(a) Evolution of the joint angle q_1



(b) Evolution of the joint angle q_2

Fig. 2: Simulation results of Example 1.

Example 2: Consider the longitudinal dynamics of a missile given in [16]:

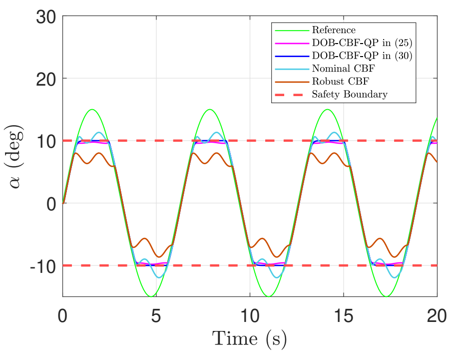
$$\dot{\alpha} = f_1(\alpha) + q + b_1(\alpha)\delta + d_1, \qquad (31a)$$

$$\dot{q} = f_2(\alpha) + b_2\delta + d_2, \qquad (31b)$$

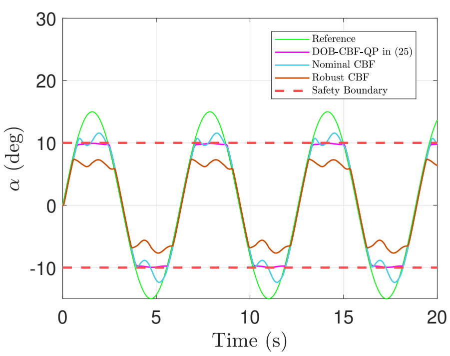
$$\dot{\delta} = (1/t_1)(-\delta + u), \qquad (31c)$$

where α is the angle of attack, q is the pitch rate, δ is the tail fin deflection, u is the control input, d_1, d_2 are mismatched disturbances satisfying $\|d_1\| \leq 30$, $\|d_2\| \leq 30$, $\|\dot{d}_1\| \leq 50$, $\|\dot{d}_2\| \leq 50$, and $f_1(\alpha)$, $b_1(\alpha)$, $f_2(\alpha)$ are nonlinear known functions given in [16]. The same bounds above are used for the robust CBF approach. It is obvious that the disturbance relative degree of d_1 is lower than the input relative degree. Although system (31) is slightly different from (16), the control approaches presented in Section III-B can be easily generalized to deal with such a system. The nominal controller is the DOB-based tracking controller given in [16].

To avoid the stall, the proposed DOB-CBF-QP-based controllers are employed to restrict the value of α in the presence of disturbances. We consider two scenarios: 1) Two CBFs are selected as $h_1 = 10-\alpha$ and $h_2 = \alpha+10$; in this case, the DOB-CBF-QP-based controllers both in (25) and (30) are applicable. 2) A single quadratic CBF $h = 100-\alpha^2$ is employed; in this case, only the DOB-CBF-QP-based controller in (25) is applicable. From the simulation results shown in Fig. 3, it can be seen that the proposed safe controllers can both ensure safety (i.e., the trajectory of α is always inside the safe regions bounded by the dashed red lines), and the trajectories of the closed-loop system are less conservative than the robust CBF approach because of the better tracking performance inside the safe region. Meanwhile, it can be observed that the DOB-CBF-QP-based controller obtained from (30) has a better tracking performance than that from (25).



(a) Evolution of the angle of attack α by using two CBFs $h_1 = 10 - \alpha$ and $h_2 = \alpha + 10$



(b) Evolution of the angle of attack α by using a single CBF $h = 100 - \alpha^2$

Fig. 3: Simulation results of Example 2.

V. Conclusion

In this paper, a new DOB-CBF-QP-based safe control design approach was proposed for systems with external disturbances, with the goal of achieving a better tradeoff between safety and performance. The simulation results demonstrate the superiority of the proposed control scheme over existing worst-case-based robust CBF techniques.

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