On Converse Zeroing Barrier Functions

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Abstract

The paper studies the safety verification problem for nonlinear systems and focuses on the converse problem of zeroing barrier functions (ZBFs). We establish two necessary and sufficient conditions for the existence of a ZBF by solving the converse ZBF problem. Moreover, we also consider exponential barrier functions (EBFs), a special case of the ZBF, and provide a necessary and sufficient condition for the existence of an EBF when the state trajectory, starting from the interior of the safe set, cannot visit the boundary within finite time.

Key words: Safety; converse theorem; zeroing barrier function; exponential barrier function.

1 Introduction

Safety is crucial for modern control systems [1]. Over the past two decades, barrier functions have emerged as a promising technique for safety analysis and control [2–6]. Barrier functions are Lyapunov-like functions. The essential idea behind barrier function approaches is to impose Lyapunov-like constraints on the "change" of the state trajectory such that the state always stays in a safe set. In [3, 7], a novel barrier function called the zeroing barrier function (ZBF) was proposed. The term "zeroing" refers to that the barrier function vanishes as the state reaches the boundary of the safe set. Compared with the barrier certificate conditions in [2], the ZBF is less conservative inside the safe set since it only requires a single super- or sub-level set to be forward invariant. In [3, 7], it was shown that the existence of a ZBF is a sufficient condition for the forward invariance and asymptotic stability of a safe set. Invariance means that the state starting from the safe set will always remain safe, while asymptotic stability implies that a state trajectory initialized outside the safe set will eventually reach the safe set. Meanwhile, ZBFs have been widely employed in safety-critical control. For the case with a single zeroing control barrier function (ZCBF), [3, 7] showed that the safety-critical controller synthesized by a ZCBF quadratic program is locally Lipschitz when the decaying rate of the ZCBF is locally Lipschitz. Recently, numerous variants of the ZBF have been reported; see, e.g., [8–15].

The exponential barrier function (EBF) is a special case of the ZBF with a linear decaying rate. With this concept, one can easily handle high-relative-degree safety constraints using linear control tools [8, 9, 34]. Over the past several years, EBFs have been widely used in practical safety control problems [17–20]. Compared with the high-relative-degree ZBFs [11, 13] with a nonlinear decaying rate, the set of EBFs has been shown to be convex [16], implying that one can use convex optimization to synthesize an EBF. For example, sum of squares programming algorithms were developed in [9, 16, 17] to compute EBFs for polynomial nonlinear systems.

Note that all the aforementioned safety analysis and control methods rely on the knowledge of a barrier function. However, there is no general technique for constructing such a function for nonlinear systems. Therefore, it is crucial to confirm the existence of a barrier function be-

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fore attempting to find one; otherwise, the search will fail if the desired barrier function does not exist. The answer to this question is provided by the so-called converse barrier function theorems.

Motivated by the significance of ZBFs and EBFs in safety analysis and control, this paper concentrates on their converse problems. Specifically, given a safe system, we aim to address the following questions:

- Does the system have a ZBF?
- Does the system have an EBF?

Over the past two decades, converse barrier function problems have been studied in [3,21–26]. In [23], the time required for the state to reach the boundary of the safe set was utilized to prove the existence of a barrier certificate. This approach does not require any Slater-like conditions used in [21] or Morse-Smale vector field conditions used in [22]. However, if the state does not reach the boundary of the safe set, the converse barrier certificate therein always equates to infinity inside the safe set. In [24,25], a Lyapunov-like analysis for set stability was used to establish converse barrier certificate theorems. In fact, these converse barrier certificates are constant inside the safe set. In [3], it is shown that, if the safe set $\mathcal{S} = \{x \in \mathbb{R}^n : h(x) \geq 0\}$ is both compact and contractive, where $h(x)$ is a continuously differentiable function capturing the boundary of the safe set, then such a function $h(x)$ is a converse ZBF. In practical applications, many control systems involve noncompact safe sets. An example is the adaptive cruise control problem studied in [3, 7, 27], where autonomous vehicles are required to maintain a safe distance. Additionally, all these works have not investigated the converse EBF problem.

This paper aims at addressing the aforementioned converse ZBF and EBF problems. In contrast to the most relevant converse barrier certificate results in [23–26], our converse barrier functions are neither infinite nor constant. Compared with the converse ZBF result in [3], our safe set is not required to be compact. Our contributions are as follows:

- We develop two converse theorems for ZBFs. In the first converse ZBF theorem (Theorem 1), we show that the existence of a ZBF with a local Lipschitz decaying rate is equivalent to that the safe set is forward invariant and the state trajectory cannot reach the boundary of the safe set within finite time. In the second converse ZBF theorem (Theorem 2), we show that the existence of a ZBF, with the corresponding decaying rate not necessarily locally Lipschitz, is equivalent to that the safe set is forward invariant and asymptotically stable.
- Regarding the converse EBF problem, we first reveal the connection between EBFs and ZBFs, and then establish a converse EBF theorem (Theorem 3) to show that the existence of an EBF is equivalent to that the safe set is forward invariant and the state trajectory cannot reach the boundary of the safe set.

Notations. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. Given a set $\mathcal{S}$, denote by $\mathrm{Int}(\mathcal{S})$, $\mathrm{cl}(\mathcal{S})$ and $\partial\mathcal{S}$ the interior, the closure and the boundary of $\mathcal{S}$, respectively. For any x in Euclidean space, $|x|$ is its norm, and $|x|_{\mathcal{S}} = \inf_{x^* \in \mathcal{S}} |x - x^*|$ denotes the point-to-set distance from x to the set $\mathcal{S}$. A continuous function γ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\gamma(0) = 0$ is of class K ($\gamma \in K$), if it is strictly increasing. A function $\gamma \in K$ is of class K_∞ ($\gamma \in K_\infty$) if $\gamma(s) \to +\infty$ as $s \to +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class KL ($\beta \in KL$), if for each fixed $t \geq 0$, the mapping $s \mapsto \beta(s,t)$ is of class K, and for fixed $s > 0$, $t \mapsto \beta(s,t)$ is decreasing to zero as $t \to +\infty$. A continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(0) = 0$ is of extended class K ($\gamma \in EK$) if it is strictly increasing. In particular, a function $\gamma \in EK$ is of extended class K_∞ ($\gamma \in EK_\infty$) if $\gamma(s) \to +\infty$ as $s \to +\infty$ and $\gamma(s) \to -\infty$ as $s \to -\infty$.

2 Preliminaries

Consider the system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field. Denote by $x(t, x_0)$ the solution of system (1) with $x(0) = x_0$. Let $(-T^-_{x_0}, T^+_{x_0})$ be the maximal time interval of existence of solutions for some $-\infty \leq -T^-_{x_0} < 0 < T^+_{x_0} \leq +\infty$. We say that system (1) is forward (or backward) complete if $T^+_{x_0} = +\infty$ (or $T^-_{x_0} = +\infty$). Moreover, system (1) is complete if it is both forward and backward complete. A set $\mathcal{S}$ is forward invariant if $x(t, x_0) \in \mathcal{S}$ for all $x_0 \in \mathcal{S}$.

Remark 1 *Different from most existing works (for example, see [3, 7]), which assumed $f(x)$ to be locally Lipschitz, we only require that it is continuous. This modification allows our results to encompass safe systems whose state trajectories may reach the boundary of the corresponding safe set within finite time. A typical example is the system under the finite-time convergent barrier function condition of [35, 36].*

Denote by $\mathcal{X}_u$ the unsafe set of system (1). Let $\mathcal{S}$ be a closed set such that $\mathcal{S} \bigcap \mathcal{X}_u = \emptyset$.

Definition 1 (Safety) *System (1) is safe on a set $\mathcal{S}$ if $\mathcal{S}$ is forward invariant.*

In the barrier function literature [2,3,7], the set $\mathcal{S}$ is characterized by a continuously differentiable scalar function

$h(x)$ as follows:

$$\mathcal{S} = \{x \in \mathbb{R}^n : h(x) \geq 0\}, \tag{2a}$$
$$\partial\mathcal{S} = \{x \in \mathbb{R}^n : h(x) = 0\}, \tag{2b}$$
$$\mathrm{Int}(\mathcal{S}) = \{x \in \mathbb{R}^n : h(x) > 0\}. \tag{2c}$$

Definition 2 *A continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ is a barrier function candidate if it is such that the set $\mathcal{S}$ in (2) is nonempty and does not intersect the unsafe region $\mathcal{X}_u$. Herein, $\mathcal{D}$ is a set such that $\mathcal{S} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$.*

Remark 2 *Condition (2) implies that $h(x)$ is zero on the boundary $\partial\mathcal{S}$, strictly positive within the interior Int($\mathcal{S}$), and strictly negative in the exterior $\mathbb{R}^n \backslash \mathcal{S}$. This distinguishes barrier functions from set Lyapunov functions (see, e.g., [32, Definition 2.6]), which are always equal to zero inside the corresponding invariant set. Thanks to (2), one can check whether the state trajectory is on the boundary by detecting the value of $h(x)$, which is particularly useful for safety-critical control because it enables one to adjust the control input to ensure safety based on both the value and the change of $h(x)$. Moreover, condition (2) implies that $h(x)$ is not allowed to be a constant. This is significant because it is easy to find counter-examples (see, e.g., [30, Example 2]) showing that a constant $h(x)$ cannot certify the forward invariance of $\mathcal{S}$ even though it satisfies a certain barrier function condition.*

It is worth mentioning that (2) can be rewritten as below.

Lemma 1 *A continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ is a barrier function candidate if and only if there exist EK-functions α_1 and α_2 such that*

$$h(x) = 0, \ \ \forall x \in \partial\mathcal{S}, \tag{3a}$$
$$\alpha_1(|x|_{cl(\mathbb{R}^n\backslash\mathcal{S})}) \leq h(x) \leq \alpha_2(|x|_{cl(\mathbb{R}^n\backslash\mathcal{S})}), \quad \forall x \in Int(\mathcal{S}), \tag{3b}$$
$$\alpha_2(-|x|_{\mathcal{S}}) \leq h(x) \leq \alpha_1(-|x|_{\mathcal{S}}), \quad \forall x \in \mathcal{D}\backslash\mathcal{S} \tag{3c}$$

The proof of of Lemma 1 is given in Appendix A.

Definition 3 (ZBF & EBF) *A barrier function candidate $h : \mathcal{D} \to \mathbb{R}$ is*

- *a zeroing barrier function (ZBF) if*[1]

$$L_f h(x) \geq -\alpha_3(h(x)), \ \ \forall x \in \mathcal{D} \tag{4}$$

  where $\alpha_3 \in EK$ is called the decaying rate function;
- *an exponential barrier function (EBF) if*

$$L_f h(x) \geq -\lambda h(x), \ \ \forall x \in \mathcal{D} \tag{5}$$

  where $\lambda > 0$ is a constant.

[1] For any differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, $L_f h(x) := \nabla V(x) f(x)$.

The ZBF condition (4) was originally proposed in [3,7], while the EBF condition (5) is a special case of (4) with the decaying rate being locally Lipschitz.

Lemma 2 ([3,7]) *System (1) is safe on the set $\mathcal{S}$ if either of the following holds:*

i) there is a ZBF satisfying (2) and (4) with the corresponding decaying rate α_3 being locally Lipschitz;
ii) there is a ZBF satisfying (2) and (4) with a given set $\mathcal{D}$ such that $\mathcal{S} \subset \mathcal{D}$.

As in Lemma 2, the existence of a ZBF is a sufficient condition for safety, when the corresponding decaying rate α_3 is locally Lipschitz, or the domain $\mathcal{D}$, where a ZBF is defined, is a set such that $\mathcal{S} \subset \mathcal{D}$. A natural question is whether these conclusions remain true when α_3 is non-Lipschitz and $\mathcal{D}$ is a set such that $\mathcal{S} = \mathcal{D}$. The answer is negative. Below is a counter-example.

Example 1 *Consider the system*

$$\dot{x} = -\sqrt{|x|}, \ \ x(0) = x_0. \tag{6}$$

The safe set $\mathcal{S} = \{x \geq 0\}$. By taking $h(x) = x$, we have $L_f h(x) = -\sqrt{|h(x)|}$ for all $x \in \mathcal{S}$. Thus, the ZBF condition given in (2) and (4) is satisfied with α_3 being non-Lipschitz and $\mathcal{D} = \mathcal{S}$. For each $x_0 \in Int(\mathcal{S})$, the solution of (6) is

$$x(t, x_0) = \begin{cases} \left(\sqrt{x_0} - \frac{1}{2}t\right)^2, & 0 \leq t \leq 2\sqrt{x_0} \\ 0, & 2\sqrt{x_0} < t \leq 2\sqrt{x_0} + \tau \\ -\frac{1}{4}(t - 2\sqrt{x_0} - \tau)^2, & t \geq 2\sqrt{x_0} + \tau \end{cases}$$

for any $\tau \geq 0$, implying that only the solution trajectory

$$x(t, x_0) = \begin{cases} \left(\sqrt{x_0} - \frac{1}{2}t\right)^2, & 0 \leq t \leq 2\sqrt{x_0} \\ 0, & t > 2\sqrt{x_0} \end{cases}$$

remains in $\mathcal{S}$, while there are infinite solution trajectories leave $\mathcal{S}$. Hence, system (6) is unsafe. The reason behind this counter-example is that the solution of (6) starting from $\partial\mathcal{S}$ is nonunique in forward time and $\mathcal{S}$ is an unstable set of (6), which results in that multiple solution trajectories continue to move outside the safe set after reaching $\partial\mathcal{S}$.

In safety analysis and control, set asymptotical stability is another important notion to characterize the robustness of a safe system against uncertainties in the initial condition.

Definition 4 (Set Asymptotic Stability [28]) *A closed and forward invariant set $\mathcal{S}$ is said to be asymptotically stable for the forward complete system (1) if* *there exists $\beta \in KL$ such that*

$$|x(t, x_0)|_{\mathcal{S}} \leq \beta(|x_0|_{\mathcal{S}}, t), \;\; \forall x_0 \in \mathcal{D}, \; \forall t \geq 0. \qquad (7)$$

In stability analysis, the set $\mathcal{D}$ in (7) is called the domain of attraction. For convenience, we say that system (1) is asymptotically stable with respect to $\mathcal{S}$ if $\mathcal{S}$ is asymptotically stable. The asymptotic stability of a safe set $\mathcal{S}$ implies that, if the system is initialized outside $\mathcal{S}$ because of uncertainties, then i) the distance between $x(t)$ and $\mathcal{S}$ is bounded by $\beta(|x_0|_{\mathcal{S}}, 0)$, and ii) $x(t)$ reaches $\mathcal{S}$ eventually.

Lemma 3 ([3, 7]) *System (1) is asymptotically stable with respect to $\mathcal{S}$ if there is a ZBF satisfying (2) and (4) with $\mathcal{D}$ being a set such that $\mathcal{S} \subset \mathcal{D}$.*

In [3, 7], it was shown that a locally Lipschitz decaying rate function α_3 is helpful for the synthesis of a locally Lipschitz safety-critical controller, but a safe system may not necessarily have a ZBF with a locally Lipschitz rate (see Example 2 in Section 3.1). To study when the decaying rate is locally Lipschitz, we have the following stronger definition of safety.

Definition 5 (Strong Safety) *System (1) is strongly safe on $\mathcal{S}$, if it is safe on $\mathcal{S}$ and there is no finite time $T \geq 0$ such that $x(T, x_0) \in \partial\mathcal{S}$ for all $x_0 \in Int(\mathcal{S})$.*

Compared with Definition 1, the strong safety additionally requires that the state starting from $\mathrm{Int}(\mathcal{S})$ cannot visit $\partial\mathcal{S}$ for all $t < +\infty$. In fact, the notion of strong safety is not new, since many existing barrier functions can guarantee strong safety, e.g., the reciprocal barrier functions of [27], the ZBFs of [7] with a locally Lipschitz decaying rate, and the EBFs of [8, 9, 16]. Below is a straightforward extension of the results in [3, 7–9, 16].

Lemma 4 *System (1) is strongly safe on the set $\mathcal{S}$ if either of the following holds:*

i) there is a ZBF satisfying (2) and (4) with a locally Lipschitz decaying rate α_3;
ii) there is an EBF satisfying (2) and (5).

The proof of Lemma 4 is given in Appendix B. Note that the aforementioned lemmas are sufficiency results. The main purpose of this paper is to investigate the necessity.

3 Main Results

This section presents three converse barrier function theorems to establish the connection between the existence of a ZBF or an EBF and the safety of a nonlinear system.

3.1 Zeroing Barrier Functions

We first consider the converse ZBF problem for the strong safety case.

Theorem 1 *Suppose that system (1) is forward complete and there is a closed set $\mathcal{S}$ such that $\mathcal{S} \bigcap \mathcal{X}_u = \emptyset$. Then it is strongly safe on $\mathcal{S}$ if and only if there is a ZBF satisfying (2) and (4) with the corresponding decaying rate being locally Lipschitz.*

The following lemma, motivated by [33, Corollary 2.4], is important for proving Theorem 1.

Lemma 5 *System (1) is strongly safe on $\mathcal{S}$ if and only if there exist functions $\chi_1, \chi_2 \in K_\infty$ and a constant $c \geq 0$ such that the solution $x(t, x_0)$ starting from any $x_0 \in Int(\mathcal{S})$ satisfies*

$$\frac{1}{|x(t, x_0)|_{cl(\mathbb{R}^n \setminus \mathcal{S})}} \leq \chi_1(t) + \chi_2\left(\frac{1}{|x_0|_{cl(\mathbb{R}^n \setminus \mathcal{S})}}\right) + c \quad (8)$$

for all $t \geq 0$.

The proof of Lemma 5 is given in Appendix C. Now, we are ready to prove Theorem 1.

Proof of Theorem 1. The sufficiency follows from Lemma 4 (i). To prove the necessity, let $\phi(\tau, x)$ be the solution of (1) passing through $x \in \mathbb{R}^n$ at $\tau = 0$. Let

$$W(t, x) = \inf_{-t \leq \tau \leq 0} \frac{1}{|\phi(\tau, x)|_{\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})}}, \;\; \forall x \in \mathrm{Int}(\mathcal{S}). \quad (9)$$

Clearly,

$$0 \leq W(t, x) \leq \frac{1}{|x|_{\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})}}. \qquad (10)$$

Let $\tau \in [-t, 0]$ be the time instant such that

$$\frac{1}{|\phi(\tau, x)|_{\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})}} = W(t, x).$$

Because system (1) strongly safe on $\mathcal{S}$, it follows from Lemma 5 that

$$\begin{aligned} \frac{1}{|x|_{\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})}} &= \frac{1}{|\phi(-\tau, \phi(\tau, x))|_{\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})}} \\ &\leq \chi_1(-\tau) + \chi_2\Big(\frac{1}{|\phi(\tau, x)|_{\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})}}\Big) + c \\ &\leq \chi_1(t) + \chi_2(W(t, x)) + c. \end{aligned}$$

Thus,

$$\chi_1^{-1}\Big(\frac{1}{2|x|_{\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})}}\Big) \leq t + \chi_1^{-1}(\chi_2(W(t, x))) + c$$

which gives

$$\eta\Big(\frac{1}{|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}\Big)\,\mathrm{e}^{-t/2}\leq\alpha(W(t,x)),\ \ \forall x\in\mathrm{Int}(\mathcal{S})\tag{11}$$

with

$$\eta(s)=\exp\Big[\frac{\chi_1^{-1}(s/2)}{2}\Big],\ \ \alpha(s)=\exp\Big[\frac{\chi_1^{-1}(\chi_2(s)+c)}{2}\Big].$$

Let

$$\tilde{W}(x)=\inf_{t\geq 0}\alpha(W(t,x))\,\mathrm{e}^t,\ \ \forall x\in\mathrm{Int}(\mathcal{S}).\tag{12}$$

By (10)-(12), we have

$$\eta\Big(\frac{1}{|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}\Big)\leq\tilde{W}(x)\leq\alpha\Big(\frac{1}{|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}\Big).\tag{13}$$

Denote the upper Dini derivative of $\tilde{W}(x)$ along the solution of system (1) by

$$D^+\tilde{W}(x)|_{(1)}=\limsup_{\varepsilon\to 0^+}\frac{\tilde{W}(\phi(\varepsilon,x))-\tilde{W}(x)}{\varepsilon}.$$

With the definition of $\tilde{W}(x)$,

$$\begin{aligned}&D^+\tilde{W}(x)|_{(1)}\\&=\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big\{\inf_{t\geq 0}\alpha(W(t,\phi(\varepsilon,x)))\,\mathrm{e}^t-\inf_{t\geq 0}\alpha(W(t,x))\,\mathrm{e}^t\Big\}\\&\leq\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big\{\inf_{t\geq \varepsilon}\alpha(W(t,\phi(\varepsilon,x)))\,\mathrm{e}^t-\inf_{t\geq 0}\alpha(W(t,x))\,\mathrm{e}^t\Big\}\\&=\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big\{\inf_{t\geq 0}\alpha(W(t+\varepsilon,\phi(\varepsilon,x)))\,\mathrm{e}^{t+\varepsilon}\\&\qquad\qquad-\inf_{t\geq 0}\alpha(W(t,x))\,\mathrm{e}^t\Big\}\end{aligned}$$

Because

$$\begin{aligned}&W(t+\varepsilon,\phi(\varepsilon,x))\\&\quad=\min\Big\{\inf_{-(t+\varepsilon)\leq\tau\leq-\varepsilon}\frac{1}{|\phi(\tau,\phi(\varepsilon,x))|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}},\\&\qquad\qquad\inf_{-\varepsilon\leq\tau\leq 0}\frac{1}{|\phi(\tau,\phi(\varepsilon,x))|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}\Big\}\\&\quad=\min\Big\{\inf_{-t\leq\tau\leq 0}\frac{1}{|\phi(\tau,x)|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}},\\&\qquad\qquad\inf_{-\varepsilon\leq\tau\leq 0}\frac{1}{|\phi(\tau,\phi(\varepsilon,x))|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}\Big\}\\&\quad=\min\Big\{W(t,x),\ \inf_{-\varepsilon\leq\tau\leq 0}\frac{1}{|\phi(\tau,\phi(\varepsilon,x))|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}\Big\}\\&\quad\leq W(t,x),\end{aligned}$$

we have

$$\begin{aligned}&D^+\tilde{W}(x)|_{(1)}\\&\quad\leq\limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\Big\{\inf_{t\geq 0}\alpha(W(t,x))\,\mathrm{e}^{t+\varepsilon}-\inf_{t\geq 0}\alpha(W(t,x))\,\mathrm{e}^t\Big\}\\&\quad=\inf_{t\geq 0}\alpha(W(t,x))\,\mathrm{e}^t\limsup_{\varepsilon\to 0^+}\frac{e^{\varepsilon}-1}{\varepsilon}\\&\quad=\tilde{W}(x).\end{aligned}\tag{14}$$

Let

$$U(x)=1/\tilde{W}(x).\tag{15}$$

By (13) and (15), we have

$$\sigma_1(|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})})\leq U(x)\leq\sigma_2(|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})})\tag{16}$$

for all $x\in\mathrm{Int}(\mathcal{S})$, where

$$\begin{aligned}\sigma_1(s)&=\exp\Big[-\frac{\chi_1^{-1}(\chi_2(1/s)+c)}{2}\Big],\\\sigma_2(s)&=\exp\Big[-\frac{1}{2}\chi_1^{-1}\Big(\frac{1}{2s}\Big)\Big].\end{aligned}$$

Because $\lim_{s\to 0^+}\sigma_1(s)=\lim_{s\to 0^+}\sigma_2(s)=0$, both σ_1 and σ_2 are K-functions. The combination of (14) and (15) yields

$$\tilde{W}(x)D^+U(x)|_{(1)}=-U(x)D^+\tilde{W}(x)|_{(1)}\geq -U(x)\tilde{W}(x).$$

Thus,

$$D^+U(x)|_{(1)}\geq-\sigma_3(U(x)),\ \ \forall x\in\mathrm{Int}(\mathcal{S})\tag{17}$$

where $\sigma_3(s)=s$ is a Lipschitz K-function. Let $\hat{h}(x)$ be a function such that $h(x)=U(x)$ for all $x\in\mathrm{Int}(\mathcal{S})$, and $\hat{h}(x)=0$ for all $x\in\partial\mathcal{S}$. Take $\mathcal{D}=\mathcal{S}$. From (17),

$$D^+\hat{h}(x)|_{(1)}\geq-\sigma_3(\hat{h}(x)),\ \ \forall x\in\mathcal{D}.\tag{18}$$

According to [32, Theorem B.1], there exist a continuously differentiable function $h:\mathcal{D}\to\mathbb{R}$ and continuous functions $\mu,\nu:\mathcal{D}\to\mathbb{R}_{>0}$ such that

$$|\hat{h}(x)-h(x)|<\mu(x),\ \ \forall x\in\mathcal{D}\tag{19}$$

and

$$L_fh(x)\geq-\sigma_3(h(x))-\nu(x),\ \ \forall x\in\mathcal{D}\tag{20}$$

Let $\mu(x)=\frac{|\hat{h}(x)|}{2}$. By substituting this into (19), we have $h(x)=0$ for all $x\in\partial\mathcal{S}$, and $\frac{1}{2}\hat{h}(x)\leq h(x)\leq\frac{3}{2}\hat{h}(x)$ for all $x\in\mathrm{Int}(\mathcal{S})$. Moreover, with (16) and the definition of $\hat{h}(x)$, we obtain $\hat{h}(x)>0$ for all $x\in\mathrm{Int}(\mathcal{S})$, and $\hat{h}(x)=0$

for all $x \in \partial\mathcal{S}$. Hence, $h(x)$ satisfies (2). Let $\nu(x)$ be a function such that $\nu(x) = \frac{1}{2}\sigma_3(\frac{1}{2}\hat{h}(x))$ for all $x \in \mathcal{S}$, and $\nu(x) = 0$ for all $x \in \partial\mathcal{S}$. With (20), we can then verify (4) with $\alpha_3 = \frac{3}{2}\sigma_3$. Thus, $h(x)$ is a converse ZBF satisfying (4) with α_3 being locally Lipschitz and $\mathcal{D} = \mathcal{S}$. □

Note that the decaying rate of a converse ZBF is not necessarily locally Lipschitz for a safe system. As shown in the following counter-example, it may be impossible for a safe system to have a ZBF with a locally Lipschitz decaying rate. Thus, the strong safety condition in Theorem 1 cannot be relaxed.

Example 2 *Consider system*

$$\dot{x} = -x^{1/3}, \quad x(0) = x_0. \tag{21}$$

Suppose that the unsafe region is $\mathcal{X}_u = \{x \in \mathbb{R} : x < 0\}$. *Take* $\mathcal{S} = \{x \in \mathbb{R} : x \geq 0\}$. *For any* $x_0 \in \mathcal{S}$, *the explicit solution is*

$$x(t) = \begin{cases} \left(x_0^{2/3} - \frac{2}{3}t\right)^{3/2}, & 0 \leq t \leq T^*(x_0) \\ 0, & t \geq T^*(x_0) \end{cases}$$

where $T^*(x_0) = \frac{3}{2}x_0^{2/3}$. *Thus, system (21) is safe on* $\mathcal{S}$ *and the state trajectory* $x(t)$ *reaches* $\partial\mathcal{S} = \{x \in \mathbb{R} : x = 0\}$ *at* $t = T^*(x_0)$. *Now, we show that it is impossible for system (21) to have a ZBF with a locally Lipschitz decaying rate* α_3. *Recall that, as shown in Lemma 1, (2) and (3) are equivalent. Hence, with (3) and the continuity of solution, there is a sufficiently small* $\delta > 0$ *such that* $h(x(t, x_0))$ *is strictly decreasing for all* $t \in (T^* - 2\delta, T^*)$ *and* $x_0 \in Int(\mathcal{S})$. *For any* $0 < \varepsilon < \delta$, *there is a constant* $\theta \in [0, 1]$ *such that*

$$\begin{aligned} h(x(T^* - \delta - \varepsilon)) - h(x(T^* - \delta)) \\ &= |\nabla h(\theta x(T^* - \delta - \varepsilon) + (1 - \theta)x(T^* - \delta))| \\ &\quad \times |x(T^* - \delta - \varepsilon) - x(T^* - \delta)| \\ &= |\nabla h(\theta x(T^* - \delta - \varepsilon) + (1 - \theta)x(T^* - \delta))| \\ &\quad \times |\varepsilon x(T^* - \delta)^{1/3}| \\ &:= K_{h,\delta,\varepsilon}|\varepsilon x(T^* - \delta)^{1/3}| \end{aligned} \tag{22}$$

where the first equality follows from the mean value theorem and the second one is based on that the solution is continuously differentiable. Moreover, from (3), the gradient of $h(x)$ *is nonzero in a sufficiently small neighbourhood of* $\partial\mathcal{S}$, *thereby implying* $K_{h,\delta,\varepsilon} > 0$. *Combining (4) and (22) yields*

$$\begin{aligned} \left.\frac{dh(x(t))}{dt}\right|_{t=T^*-\delta} &= \lim_{\varepsilon \to 0^+} \frac{h(x(T^* - \delta - \varepsilon)) - h(x(T^* - \delta))}{-\varepsilon} \\ &= \lim_{\varepsilon \to 0^+} \frac{K_{h,\delta,\varepsilon}|\varepsilon x(T^* - \delta)^{1/3}|}{-\varepsilon} \\ &= -K_{h,\delta,\varepsilon}|x(T^* - \delta)^{1/3}| \\ &\geq -\alpha_3(h(x(T^* - \delta))). \end{aligned} \tag{23}$$

Because α_3 *is strictly increasing and locally Lipschitz, there is a constant* $K > 0$ *such that* $|\alpha_3(h(x(T^* - \delta)))| \leq K|x(T^* - \delta)|$. *By combining this with (23), we have* $\lim_{\delta \to 0^+} |x(T^* - \delta)^{1/3}| \leq \lim_{\delta \to 0^+} K/K_{h,\delta,\varepsilon}|x(T^* - \delta)|$, *which contradicts that* $x^{1/3}$ *is non-Lipschitz at* $x = 0$. *Thus, system (21) does not have a ZBF with a locally Lipschitz decaying rate.*

Next, we consider the converse ZBF problem for safe systems whose state trajectory may reach the boundary of the safe set within finite time.

Theorem 2 *Suppose that system (1) is forward complete and there is a closed set* $\mathcal{S}$ *such that* $\mathcal{S} \bigcap \mathcal{X}_u = \emptyset$. *Then it is safe on* $\mathcal{S}$ *and asymptotically stable with respect to* $\mathcal{S}$ *if and only if there is a ZBF satisfying (3) and (4) with* $\mathcal{D}$ *being a set such that* $\mathcal{S} \subset \mathcal{D}$.

Proof. The sufficiency follows from Lemma 2 and Lemma 3. To prove the necessity, let $\phi(\tau, x)$ be the solution of (1) passing through $x \in \mathbb{R}^n$ at $\tau = 0$. If system (1) is safe, we have the following two cases for all $x \in \text{Int}(\mathcal{S})$:

- infinite-time reachability: there is no finite time instant $\tau \geq 0$ such that $\phi(\tau, x) \in \partial\mathcal{S}$;
- finite-time reachability: there is a finite time instant $\tau \geq 0$ such that $\phi(\tau, x) \in \partial\mathcal{S}$.

For the first case, it follows from the proof of Theorem 1 that there is a local Lipschitz $U(x)$ satisfying (16) and (17) for all $x \in \text{Int}(\mathcal{S})$. Additionally, because system (1) is forward complete and asymptotically stable with respect to the safe set $\mathcal{S}$, it follows from [29, Theorem 4.17] that there is a continuously differentiable Lyapunov function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ such that

$$\sigma_4(|x|_{\mathcal{S}}) \leq V(x) \leq \sigma_5(|x|_{\mathcal{S}}), \tag{24}$$

$$L_f V(x) \leq -\sigma_6(V(x)) \tag{25}$$

for all $x \in \mathcal{D}$, where $\mathcal{D}$ is the domain of attraction containing $\mathcal{S}$, σ_4, σ_5 are K_∞-functions, and σ_6 is a K-function. Moreover, σ_6 is locally Lipschitz; otherwise, one can replace it by a locally Lipschitz function $\bar{\sigma}_6(s) = \max\{s, \sigma_6(s)\}$. Let $\hat{h}(x)$ be a function such that $\hat{h}(x) = U(x)$ for all $x \in \text{Int}(\mathcal{S})$, $\hat{h}(x) = 0$ for all $x \in \partial\mathcal{S}$, and $\hat{h}(x) = -V(x)$ for all $x \in \mathcal{D}\backslash\mathcal{S}$. By combining (16), (17), (24), and (25), we have

$$\hat{h}(x) = 0, \quad \forall x \in \partial\mathcal{S}, \tag{26a}$$

$$\kappa_1(|x|_{\text{cl}(\mathbb{R}^n\backslash\mathcal{S})}) \leq \hat{h}(x) \leq \kappa_2(|x|_{\text{cl}(\mathbb{R}^n\backslash\mathcal{S})}), \quad \forall x \in \text{Int}(\mathcal{S}), \tag{26b}$$

$$\kappa_2(-|x|_{\mathcal{S}}) \leq \hat{h}(x) \leq \kappa_1(-|x|_{\mathcal{S}}), \quad \forall x \in \mathcal{D}\backslash\mathcal{S} \tag{26c}$$

and

$$D^+\hat{h}(x)|_{(1)} \geq -\kappa_3(\hat{h}(x)), \quad \forall x \in \mathcal{D} \tag{27}$$

where, for $i = 1, 2, 3$, κ_i is a function such that $\kappa_i(0) = 0$, $\kappa_i(s) = \sigma_i(s)$ for all $s > 0$, and $\kappa_i(s) = -\sigma_{3+i}(-s)$ for all $s < 0$. By applying [32, Theorem B.1] to (26) and (27), we obtain that there exist a continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ satisfying (19) and (20). Take $\mu(x) = \frac{|\hat{h}(x)|}{2}$. Let $\nu(x)$ be a function such that $\nu(x) = \frac{1}{2}\kappa_3(\frac{1}{2}\hat{h}(x))$ for all $x \in \mathcal{S}$, $\nu(x) = 0$ for all $x \in \partial\mathcal{S}$, and $\nu(x) = \frac{1}{2}\kappa_3(\frac{3}{2}\hat{h}(x))$ for all $x \in \mathcal{D}\backslash\mathcal{S}$. Then we can verify that such a function $h(x)$ is a converse ZBF satisfying (2) and (4).

Next, we consider the second case. Let

$$T(x) = \inf\{t \geq 0 : \phi(t, x) \in \partial\mathcal{S}\} \tag{28}$$

be the first time that the state trajectory $\phi(t, x)$ starting from any $x \in \mathrm{Int}(\mathcal{S})$ reaches $\partial\mathcal{S}$. Clearly, $T(x)$ is zero on $\partial\mathcal{S}$ and continuous on $\mathcal{S}$. By the definition of $T(x)$ in (28) and the continuity of $\phi(t, x)$, there is a sufficiently small $\bar{t} \geq 0$ such that

$$T(\phi(t, x)) = T(x) - t, \ \ \forall x \in \mathrm{Int}(\mathcal{S}), \ \forall t \in [0, \bar{t}]. \tag{29}$$

Let

$$U(x) = (T(x))^{\frac{1}{1-c}}, \ \ x \in \mathrm{Int}(\mathcal{S}) \tag{30}$$

where $c \in (0, 1)$ is a constant. According to [29, Lemma 4.3], because $T(x)$ vanishes on $\partial\mathcal{S}$ and is positive for all $x \in \mathrm{Int}(\mathcal{S})$, there exist functions $\sigma_1, \sigma_2 \in K$ such that (16) holds. Additionally, combining (29) and (30),

$$\begin{aligned} D^+U(x)|_{(1)} &= \frac{1}{1-c}(T(x))^{\frac{c}{1-c}} \lim_{\varepsilon \to 0^+} \frac{T(\phi(\varepsilon, x)) - T(x)}{\varepsilon} \\ &= -\frac{1}{1-c}(T(x))^{\frac{c}{1-c}}, \quad \forall x \in \mathrm{Int}(\mathcal{S}) \end{aligned} \tag{31}$$

which is identical to (17) with $\sigma_3(s) = \frac{1}{1-c}s^c$. Hence, repeating the analysis in the first case, we can verify that there is a converse ZBF $h(x)$ satisfying (2) and (4).

The proof is completed by combining the two cases above. □

Remark 3 *Theorem 2 differs from Theorem 1 in both sufficient and necessary parts. For sufficiency, Theorem 2 does not require the decaying rate of a ZBF to be locally Lipschitz. For necessity, different from the strong safety condition in Theorem 1, Theorem 2 considers a more general safety property, where the state trajectory may reach the boundary of the safe set.*

Remark 4 *Converse theorems are fundamental for safety analysis and control based on barrier functions, as they address the existence problem of barrier functions, ensuring that the search for a barrier function is not hopeless. The most relevant works of our converse ZBF theorems are [23–26], where converse barrier function theorems were developed for the system*

$$\dot{x} = f(x) + d, \ \ \forall x_0 \in \mathcal{X}_0, \ \ \forall |d| \leq \delta \tag{32}$$

with $\delta \geq 0$. In [23], the backward time $T^*_{\leq 0}(x) = \sup\{t \leq 0 : \phi(t, x) \in \partial\mathcal{S}\}$ required for the state trajectory $\phi(t, x)$ starting from any $x \in Int(\mathcal{S})$ to reach $\partial\mathcal{S}$ was used to construct a converse barrier certificate. Therein, it was shown that $h(x) = -T^*_{\leq 0}(x)$ is a converse barrier certificate such that $\nabla h(x) f(x) > 0$ for all $x \in \partial\mathcal{S}$ and $|d| \leq \delta$. With a similar idea, a converse barrier certificate theorem was proposed in [26] for differential inclusions. However, if the state trajectory cannot reach $\partial\mathcal{S}$, the reachable time will be infinite, and thus, the corresponding converse barrier certificate will be always equal to infinity. In [25, Theorem 16], it was shown that, if $\mathcal{S}$ is a forward invariant set for system (32) under disturbance $|d| \leq \delta$, then such a set is an asymptotically stable set for the system under smaller disturbance $|\bar{d}| < \delta$. Together with the converse Lyapunov theorem, it was shown that the function $h(x) = c - V(x)$ is a converse barrier certificate such that $\nabla h(x)(f(x) + \bar{d}) > 0$ for all $x \in \mathbb{R}^n$ and $|\bar{d}| < \delta$, where $c > 0$ is a constant and $V(x)$ is a converse Lyapunov function with respect to $\mathcal{S}$. In [24, Theorem 2], it was shown that $h(t, x) = \sup_{-t \leq \tau \leq 0} |\phi(\tau, x)|_{\mathcal{S}}$ is a time-varying converse barrier certificate for system (32) such that $\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}[f(x) + d] \geq 0$ for all $x \in \mathbb{R}^n$ and $|d| \leq \delta$. It should be noted that both converse barrier certificates of [26] and [25] are constant when the state is inside the invariant safe set, which implies that these barrier certificates do not satisfy the barrier function candidate condition given in (2). Compared with [23–26], our converse ZBF is neither equal to infinity nor equal to a constant. The value of our converse ZBF increases with the distance between the state and the boundary of the safe set.

Now, we provide an example for the construction of a converse ZBF when the state trajectory $x(t, x_0)$ starting from any $x_0 \in \mathrm{Int}(\mathcal{S})$ cannot reach $\partial\mathcal{S}$.

Example 3 *Consider the system*

$$\dot{x} = -x^3, \ \ x(0) = x_0. \tag{33}$$

Suppose that the unsafe region of system (33) is $\mathcal{X}_u = \{x \in \mathbb{R} : x < 0\}$. Take $\mathcal{S} = \{x \in \mathbb{R} : x \geq 0\}$. For any $x_0 \in \mathcal{S}$, the solution of (33) is $x(t, x_0) = x_0/\sqrt{1 + 2x_0^2 t}$, which implies $1/|x(t, x_0)| \leq 2\sqrt{t} + \sqrt{2}/|x_0| := \chi_1(t) + \chi_2(1/|x_0|)$. With (9), $W(t, x) = \sqrt{-2t + 1/x^2}$. Take $\alpha(s) = \exp(s^2/4)$. By (12), $\tilde{W}(x) = \inf_{t \geq 0} \alpha(W(t, x)) \exp(t) = \exp(1/4x^2)$. Hence, according to (15), $U(x) = 1/\tilde{W}(x) = \exp(-1/4x^2)$. Clearly, such a function $U(x)$ satisfies the ZBF condition (4) for all $x \in \mathcal{S}$.

Then we show the construction of a converse ZBF when the state trajectory $x(t, x_0)$ starting from any $x_0 \in \text{Int}(\mathcal{S})$ reaches $\partial\mathcal{S}$ within finite time.

Example 4 *Consider system (21) again. By (30), we have $U(x) = (\frac{3}{2})^{\frac{3}{2}} x$. Clearly, $U(x)$ also satisfies (2) and (4) for all $x \in \mathcal{S}$.*

3.2 Exponential Barrier Functions

To prove the existence of an EBF, we introduce the following lemma to connect ZBFs with EBFs. This lemma is motivated by [31, Proposition 13] and [32, Theorem 2.8].

Lemma 6 *If $h : \mathcal{D} \to \mathbb{R}$ is a ZBF satisfying (2) and (4) for some locally Lipschitz EK-function α_3, then there exists a continuously differentiable EK-function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\hbar(x) = \rho(h(x))$ is an EBF.*

Proof. Because α_3 is locally Lipschitz and strictly increasing, we can find a continuously differentiable function $\alpha \in EK$ such that $|\alpha(s)| \geq |\alpha_3(s)|$ for all $s \geq 0$. Let

$$\rho(s) = \begin{cases} \exp\Big(\int_1^s \frac{\lambda \mathrm{d}r}{\alpha(r)}\Big), & s > 0; \\ 0, & s = 0; \\ -\exp\Big(\int_{-1}^s \frac{\lambda \mathrm{d}r}{\alpha(r)}\Big), & s < 0. \end{cases}$$

Now, we are ready to show that i) ρ is an EK-function; ii) ρ is continuously differentiable on $\mathbb{R}$; and iii) $\rho(h(x))$ is an EBF as desired.

Step 1: Proving $\rho \in EK$. Clearly, $\rho(s)$ is strictly increasing on $\mathbb{R}$ and continuous on $\mathbb{R}\backslash\{0\}$. The rest is to show that $\rho(s)$ is continuous at $s = 0$. Because α is continuously differentiable and strictly increasing on $\mathbb{R}$, there exists a constant $K > 0$ such that $|\alpha(s)| \leq K|s|$ for all $s \in [-1, 1]$. Thus,

$$\rho(s) \leq \exp\Big(\int_1^s \frac{\lambda \mathrm{d}r}{Kr}\Big) = s^{\frac{\lambda}{K}}, \ \ \forall s \in [0, 1], \tag{34a}$$

$$\rho(s) \geq -\exp\Big(\int_{-1}^s \frac{\lambda \mathrm{d}r}{Kr}\Big) = -s^{\frac{\lambda}{K}}, \ \ \forall s \in [-1, 0). \tag{34b}$$

Hence, $\rho(r)$ is continuous at $r = 0$, and thus, is an EK-function.

Step 2: Proving the differentiability of ρ. Because α is strictly increasing and continuously differentiable on $\mathbb{R}$, there is a constant $K > 0$ such that

$$0 < \alpha'(s) = \lim_{\delta \to 0} \frac{\alpha(s + \delta) - \alpha(s)}{\delta} \leq K, \ \ \forall s \in [-1, 1].$$

Again with the differentiability of α, we get that ρ is C^2 on $\mathbb{R}\backslash\{0\}$ and satisfies

$$\rho'(s) = \frac{\lambda\rho(s)}{\alpha(s)}, \ \ \rho''(s) = \frac{\lambda\rho(s)}{\alpha^2(s)}\Big(\lambda - \alpha'(s)\Big), \ \ \forall s \neq 0.$$

Take $\lambda > K$, and then,

$$\rho''(s) \geq \frac{\lambda\rho(s)}{\alpha^2(s)}\Big(\lambda - K\Big) > 0, \ \ \forall s \in [-1, 0) \cup (0, 1],$$

which implies that $\rho'(s)$ is strictly increasing on $[-1, 0) \cup (0, 1]$. On the other hand, from (34),

$$\lim_{s \to 0^+} \frac{\rho(s) - \rho(0)}{s} \leq s^{\frac{\lambda - K}{K}}, \ \ \lim_{s \to 0^-} \frac{\rho(s) - \rho(0)}{s} \geq -s^{\frac{\lambda - K}{K}},$$

which implies $\rho'(0) = 0$. In the following, we show

$$\lim_{s \to 0^+} \rho'(s) = \lim_{s \to 0^-} \rho'(s) = \rho'(0) = 0$$

by contradiction. Because $\rho(s)$ is strictly increasing, $\rho'(s)$ is positive on $\mathbb{R}\backslash\{0\}$, and therefore, there exists $c > 0$ such that $\rho'(s) \geq c$ for all $s \in \mathbb{R}\backslash\{0\}$. Thus, for any $s \in (0, 1]$,

$$\rho''(s) = \frac{\lambda\rho(s)}{\alpha^2(s)}\Big(\lambda - \alpha'(s)\Big) = \frac{\rho'(s)}{\alpha(s)}\Big(\lambda - \alpha'(s)\Big) \geq \frac{c(\lambda - K)}{Ks}.$$

Hence,

$$\begin{aligned} \lim_{s \to 0^+} \rho'(s) &= \rho'(1) - \lim_{s \to 0^+} \int_s^1 \rho''(r) dr \\ &\leq \rho'(1) - \frac{c(\lambda - K)}{K} \lim_{s \to 0^+} \int_s^1 \frac{dr}{r} \\ &= -\infty \end{aligned}$$

which contradicts the assumption $\lim_{s \to 0^+} \rho'(s) > 0$, and thus, $\lim_{s \to 0^+} \rho'(s) = \rho'(0) = 0$. Similarly, we can show $\lim_{s \to 0^-} \rho'(s) = \rho'(0) = 0$. Thus, ρ is continuously differentiable on $\mathbb{R}$.

Step 3: Proving that $\rho(h(x))$ is an EBF. Let $\hbar(x) = \rho(h(x))$. Because ρ is an EK-function, we have $\hbar(x) = 0$ for all $x \in \partial\mathcal{S}$, $\hbar(x) > 0$ for all $x \in \text{Int}(\mathcal{S})$, and $\hbar(x) < 0$ for all $x \in \mathcal{D}\backslash\mathcal{S}$. Moreover,

$$L_f \hbar(x) = \frac{\lambda\rho(h)}{\alpha(h)} \cdot L_f h(x) \geq -\frac{\lambda\rho(h)\alpha_3(h)}{\alpha(h)} \geq -\lambda\hbar(x).$$

for all $x \in \mathcal{D}$. Hence, $\hbar(x)$ is an EBF. □

Recalling Theorem 1, we can construct a converse ZBF satisfying (2) and (4) with a locally Lipschitz decaying rate α_3, if system (1) is strongly safe on $\mathcal{S}$. By combining this with Lemma 4 (ii) and Lemma 6, we have the following converse EBF result.

Theorem 3 *Suppose that system (1) is forward complete, and there is a closed set $\mathcal{S}$ such that $\mathcal{S} \bigcap \mathcal{X}_u = \emptyset$. Then it is strongly safe on $\mathcal{S}$ if and only if there is an EBF satisfying (2) and (5).*

Because an EBF is a special ZBF with a globally Lipschitz decaying rate $\alpha_3(s) = \lambda s$, it follows from Example 2 that the strong safety condition in Theorem 3 for the existence of an EBF cannot be relaxed. Also note that the set of EBFs is convex, which is beneficial for the barrier function computation. This can be verified by taking any two EBFs $h_1(x)$ and $h_2(x)$ satisfying (2) and (5). It can then be shown that $h(x) = \theta h_1(x) + (1-\theta)h_2(x)$ is also a valid EBF for each $\theta \in [0, 1]$. Thus, we have the following corollary.

Corollary 1 *Under the conditions in Theorem 3, system (1) is strongly safe on $\mathcal{S}$ if and only if there is a convex set of barrier functions satisfying (2) and (5).*

4 Conclusion

This paper has explored the connections among ZBFs, EBFs, and safety by solving the converse barrier function problems. Two cases of safe systems have been studied: in the first category, the state trajectory starting from the interior of the safe set cannot reach the boundary; in the second category, the state trajectory may approach the boundary of the safe set within finite time, yet the safe set is asymptotically stable. We have proved that both categories of systems have a ZBF. Furthermore, by establishing the connection between ZBFs and EBFs, we have also shown that the first category of systems has an EBF.

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A Proof of Lemma 1

The implication (3) $\Rightarrow$ (2) is clear. Now, we verify the converse with a proof similar to that of [29, Lemma 4.3]. Let $\hat{\alpha}_1$ be a function such that $\hat{\alpha}_1(0) = 0$, $\hat{\alpha}_1(r) = \inf_{|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}\geq r} h(x)$ for all $r > 0$, and $\hat{\alpha}_1(r) = \sup_{|x|_{\mathcal{S}}\geq -r} h(x)$ for all $r < 0$. Let $\hat{\alpha}_2$ be a function such that $\hat{\alpha}_2(0) = 0$, $\hat{\alpha}_2(r) = \sup_{|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}\leq r} h(x)$ for all $r > 0$, and $\hat{\alpha}_2(r) = \inf_{-r\leq|x|_{\mathcal{S}}} h(x)$ for all $r < 0$. Then, we have $\hat{\alpha}_1(|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}) \leq h(x) \leq \hat{\alpha}_2(|x|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})})$ for all $x \in \mathrm{Int}(\mathcal{S})$, and $\hat{\alpha}_2(-|x|_{\mathcal{S}}) \leq h(x) \leq \hat{\alpha}_1(-|x|_{\mathcal{S}})$ for all $x \in \mathcal{D}\setminus\mathcal{S}$. By (2) and the continuity of $h(x)$, both $\hat{\alpha}_1(r)$ and $\hat{\alpha}_2(r)$ vanish at $r = 0$, and are continuous and non-decreasing for all $r \in \mathbb{R}$. Hence, (3) follows by taking EK-functions α_1 and α_2 such that $|\alpha_1(r)| \leq |\hat{\alpha}_1(r)|$ and $|\hat{\alpha}_2(r)| \leq |\alpha_2(r)|$.

B Proof of Lemma 4

Because an EBF is a special ZBF with a globally Lipschitz rate $\alpha_3(s) = \lambda s$, the second conclusion follows if we can verify the first conclusion. With Lemma 2, we obtain that the safe set $\mathcal{S}$ is forward invariant if system (1) has a ZBF satisfying (2) and (4) with a locally Lipschitz decaying rate. Now, we show that the state trajectory $x(t, x_0)$ starting from any $x_0 \in \mathrm{Int}(\mathcal{S})$ at $t = 0$ cannot reach the boundary $\partial\mathcal{S}$. Assume that this is not true. Then there exist an initial state $x_0 \in \mathrm{Int}(\mathcal{S})$ and a finite $t \geq 0$ such that $x(t, x_0) \in \partial\mathcal{S}$. Take $T^* = \inf\{0 < t < +\infty : x(t, x_0) \in \partial\mathcal{S}\}$ and $t_1 = \sup\{0 \leq t < T^* : h(x(t, x_0)) = 1\}$. Since $h(x(t, x_0))$ is continuous on t, $h(x(t, x_0)) > 0$ for all $0 \leq t < T^*$ and $h(x(t, x_0)) \leq 1$ for all $t \geq t_1$. Because α_3 is locally Lipschitz and strictly increasing, there is a constant $K > 0$ such that $\alpha_3(s) \leq Ks$ for all $0 \leq s \leq 1$. Recalling (4), we have $L_f h(x) \geq -Kh(x(t))$ for all x satisfying $0 \leq h(x) \leq 1$. Hence, $h(x(t, x_0)) \geq h(x(t_1, x_0))\,\mathrm{e}^{-K(t-t_1)} = \mathrm{e}^{-K(t-t_1)} > 0$ for all $t > t_1$. This, together with (3b), implies $|x(t, x_0)|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})} > 0$ for all $t > t_1$, contradicting $T^* < +\infty$.

C Proof of Lemma 5

We first show the necessity. Let $\gamma(t, s)$ be a function such that $\gamma(t, 0) = 0$ and

$$\gamma(t,s) = \sup\left\{\frac{1}{|x(\tau,x_0)|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}} : \tau \leq t, |x_0|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})} \geq 1/s\right\}$$

for all $s > 0$. Clearly, $\gamma(t, s)$ is positive. For any fixed $t \geq 0$ and any $s_2 > s_1 > 0$,

$$\begin{aligned}\gamma(t,s_2) = \max\Big\{&\gamma(t,s_1),\\ &\sup\left\{\frac{1}{|x(\tau,x_0)|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}} : \tau \leq t, 1/s_2 \leq |x_0|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})} < 1/s_1\right\}\Big\}\\ &\leq \gamma(t,s_1)\end{aligned}$$

which implies that $s \mapsto \gamma(t, s)$ is non-decreasing. Similarly, it is not difficult to show that $t \mapsto \gamma(t, s)$ is also non-decreasing. Let $\chi(s) = \gamma(s, s)$. Because χ is non-decreasing, there exist a function $\tilde{\chi} \in K_\infty$ and a constant $c_0 \geq 0$ such that $\chi(s) \leq \tilde{\chi}(s) + c_0$. Thus,

$$\begin{aligned}\frac{1}{|x(t,x_0)|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}} &\leq \gamma(t, \frac{1}{|x_0|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}})\\ &\leq \chi_1(t) + \chi_2(\frac{1}{|x_0|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}) + c\end{aligned}$$

where $\chi_i = \tilde{\chi}$ and $c = 2c_0$.

Next, we show the sufficiency. For any $x_0 \in \mathrm{Int}(\mathcal{S})$, $\chi_2(\frac{1}{|x_0|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})}}) < +\infty$. Thus, the term on the right-hand side of (8) cannot escape to $+\infty$ in finite time. Hence, $|x(t, x_0)|_{\mathrm{cl}(\mathbb{R}^n\setminus\mathcal{S})} > 0$ for all $t \geq 0$, which implies the strong safety of system (1).