Control Barrier Function Meets Interval Analysis: Safety-Critical Control with Measurement and Actuation Uncertainties

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Abstract— **This paper presents a framework for designing provably safe feedback controllers for sampled-data control affine systems with measurement and actuation uncertainties. Based on the interval Taylor model of nonlinear functions, a sampled-data control barrier function (CBF) condition is proposed which ensures the forward invariance of a safe set for sampled-data systems. Reachable set overapproximation and Lasserre's hierarchy of polynomial optimization are used for finding a margin term in the sampled-data CBF condition. Sufficient conditions for a safe controller in the presence of measurement and actuation uncertainties are proposed. The effectiveness of the proposed method is illustrated by a numerical example and an experimental example that implements the proposed controller on the Crazyflie quadcopter in real-time.**

I. Introduction

Designing feedback controllers that enforce safety specifications is a recurring challenge in many real systems such as automotive and robotic systems. Safety conditions are normally specified in terms of forward invariance of a set, which can be established through the barrier function (or barrier certificate) without finding trajectories of a system [1]–[3]. For control systems, controlled invariant sets are used to encode the correct behavior of the closed-loop systems and characterize a set of feedback control laws that will achieve it [4], [5]. Inspired by automotive safety-control problems, [6]–[8] proposed reciprocal and zeroing control barrier functions (CBFs) which extend previous barrier conditions to only requiring a single sub-level set be controlled invariant. Families of control policies that guarantee safety can be obtained by solving a convex quadratic program (QP). This CBF-QP framework has been used in applications such as automotive safety systems, bipedal robots, quadcopters, robotic manipulators, and multi-agent systems [9]–[13].

CBFs proposed in [6]–[8] provide forward invariance guarantees for the safety set in the continuous-time sense, but require the control law being updated continuously. This requirement is difficult to realize in practice because most controllers are digitally implemented in a sampled-data fashion and the time to solve the convex QP is not negligible in safety-critical applications. Therefore, new conditions are needed to ensure the forward invariance for sampled-data systems with piecewise-constant controllers (also called zero-order-hold controllers). CBFs for sampled-data systems have been investigated in [12], [14]–[18]. These existing works are either designed only for a specific type of systems or based on non-convex optimization which is not suitable for real-time control implementations. On the other hand, almost all the existing results using CBFs rely on accurate state and actuation information, which is difficult to obtain in practice. In [19], a measurement-robust CBF was proposed for the safety of learned perception modules; in [20], an unscented Kalman filter is integrated with CBF-QP to attenuate the effects of state disturbances and measurement noises. In spite of these interesting results, a systematic approach to handle measurement and actuation uncertainties for the CBF-based safe controller is still lacking, especially one that is suitable for real-time applications.

This paper presents a safety control design framework for sampled-data systems in the presence of measurement and actuation uncertainties, by leveraging tools from interval analysis and CBFs. The contributions are at least threefold: (i) Based on the interval Taylor model of nonlinear functions, a sampled-data control barrier function (SDCBF) condition is proposed to guarantee the forward invariance of a safe set for sampled-data systems. (ii) Sufficient conditions for CBF-based sampled-data safe controller in the presence of inaccurate state measurement and actuation are proposed. (iii) Efficient algorithms are proposed to compute the new SDCBF conditions utilizing interval arithmetic and polynomial optimization techniques, and implemented on the Crazyflie quadcopter hardware.

II. Preliminaries and Problem Statement

A. Control Barrier Functions

Consider a control affine system

$$\dot{x} = F(x, u) := f(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous. Given a control signal $u(t)$, the solution of (1) at time t with initial condition $x_0 \in \mathbb{R}^n$ at time t_0 is denoted by $x(t; t_0, x_0)$ or simply $x(t)$ when t_0, x_0 are clear from context. For simplicity, assume that the solution of (1) exists for all $t \geq t_0$. A set $\mathcal{S}$ is called (forward) controlled invariant with respect to system (1) if for every $x_0 \in \mathcal{S}$, there exists a control signal $u(t)$ such that $x(t; t_0, x_0) \in \mathcal{S}$ for all $t \geq t_0$. The set $\mathcal{S}$ is called safe if it is controlled invariant.

The (forward) *reachable set* of system (1) from an initial set $\mathcal{X}_0 \subset \mathbb{R}^n$ at time $t > t_0$ is defined as $\mathcal{R}(t, \mathcal{X}_0) \triangleq \{x(t; t_0, x_0) \mid x_0 \in \mathcal{X}_0, u \in U\}$. The (forward) *reachable tube* of system (1) from an initial set $\mathcal{X}_0$ over a time interval $[t_1, t_2]$ where $t_2 > t_1 \geq t_0$ is $\mathcal{R}([t_1, t_2], \mathcal{X}_0) \triangleq \{x(t; t_0, x_0) \mid x_0 \in \mathcal{X}_0, t \in [t_1, t_2], u \in U\}$.

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Consider a safe set $\mathcal{C} \subset \mathbb{R}^n$ defined by

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\} \tag{2}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth function. Zeroing CBFs with relative degree 1 were first proposed in [7], and generalized to CBFs with a higher relative degree in [21], [22]. In this paper, we develop our results for CBFs with a general relative degree. Given a $C^r (r \geq 1)$ function $h(x)$ with a relative degree r, it is called a zeroing CBF if there exists a column vector $\mathbf{a} \in \mathbb{R}^r$ such that $\forall x \in \mathbb{R}^n$,

$$\sup_{u \in U} [L_f^r h(x) + L_g L_f^{r-1} h(x) u + \mathbf{a}^\top \eta(x)] \geq 0 \tag{3}$$

where $\eta(x) = [L_f^{r-1} h, L_f^{r-2} h, ..., h]^\top \in \mathbb{R}^r$, and $\mathbf{a} = [a_1, ..., a_r]^\top \in \mathbb{R}^r$ is chosen such that the roots of $p_0^r(\lambda) = \lambda^r + a_1 \lambda^{r-1} + ... + a_{r-1}\lambda + a_r$ are all negative reals $-\lambda_1, ..., -\lambda_r < 0$ [21], [22]. The set of control inputs that satisfy (3) for any given $x \in \mathbb{R}^n$ is defined as $K_{\mathrm{zcbf}}(x) = \{u \in U \mid L_f^r h(x) + L_g L_f^{r-1} h(x) u + \mathbf{a}^\top \eta(x) \geq 0\}$.

Define functions $s_k(x(t))$ for $k = 0, 1, ..., r-1$ as follows:

$$s_0(x(t)) = h(x(t)), \ s_k(x(t)) = (\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_k) s_{k-1}. \tag{4}$$

It was shown in [22] that if $s_k(x(0)) \geq 0$ for $k = 0, 1, ..., r-1$, then any locally Lipschitz controller $u(x) \in \{u \in U \mid L_g L_f^{r-1} h(x) u + L_f^r h(x) + \mathbf{a}^\top \eta(x) \geq 0\}$ will guarantee the forward invariance of $\mathcal{C}$. For any given x, the safe control law is obtained by solving the following convex CBF-QP:

$$\begin{aligned} u^*(x) = \operatorname*{argmin}_{u \in U} \quad & \|u - u_{nom}\|_2 \qquad \text{(CBF-QP)} \\ \text{s.t. } & L_f^r h(x) + L_g L_f^{r-1} h(x) u + \mathbf{a}^\top \eta(x) \geq 0 \end{aligned}$$

where u_{nom} is a nominal controller that is potentially unsafe.

B. Interval Arithmetic

A real interval $[a] = [\underline{a}, \bar{a}]$ is a subset of $\mathbb{R}$. The set of all real intervals of $\mathbb{R}$ is denoted as $\mathbb{IR}$. The set of n-dimensional real interval vectors is denoted by $\mathbb{IR}^n$. Real arithmetic operations on $\mathbb{R}$ can be extended to $\mathbb{IR}$ as follows: for $\circ \in \{+, -, *, \div\}$, $[a] \circ [b] = \{\inf_{x \in [a], y \in [b]} x \circ y, \sup_{x \in [a], y \in [b]} x \circ y\}$. Classical operations for interval vectors are direct extensions of the same operations for real vectors [23], [24].

C. Problem Statement

The safety guarantee provided by the control input $u^*(x)$ from (CBF-QP) is predicated on the following implicit "assumptions": (i) the time to solve the QP is negligible so that the CBF-QP controller is updated continuously; (ii) the accurate state information is known; (iii) the exact control input generated by (CBF-QP) is implemented by the actuator. However, these "assumptions" can hardly be satisfied in reality: modern systems are predominantly based on digital electronics, which means that the input can only be updated at discrete time instances; the state information is usually obtained from sensors and contaminated with unknown noise; the desired control command is not perfectly achievable by real-life actuators.

For a sampled-data system, the sampling instants are described by a sequence of strictly increasing positive real numbers $\{t_k\}$, $k \in \mathbb{Z}_{\geq 0}$, where $t_0 = 0$, $t_{k+1} - t_k > 0$, $\lim_{k \to \infty} t_k = \infty$. Define the sampling interval between t_k and t_{k+1} as $\Delta_k = t_{k+1} - t_k$. At each sampling instance t_k, the state and input of the system are denoted as $x_k = x(t_k)$ and $u_k = u(t_k)$, respectively. The control input $u(t)$ is assumed to be a piecewise constant signal, i.e.,

$$u(t) = u_k, \ \forall t \in [t_k, t_{k+1}). \tag{5}$$

At each sampling time t_k, the control input u_k is chosen from the set $K_{\mathrm{zcbf}}(x_k)$, i.e.,

$$L_f^r h(x_k) + L_g L_f^{r-1} h(x_k) u_k + \mathbf{a}^\top \eta(x_k) \geq 0. \tag{6}$$

The CBF condition in (6) may not be satisfied during inter-sampling times $[t_k, t_{k+1})$, and therefore, it may not hold in the continuous-time sense, which means that the forward invariance of the safe set $\mathcal{C}$ may not be guaranteed.

Consider system (1) and a safe set $\mathcal{C}$ shown in (2). Suppose that the state measurement and actuation are perfect. The first problem that will be studied in this paper is stated as follows.

Problem 1: *Design a sampled-data CBF-QP controller shown in* (5) *that renders the set $\mathcal{C}$ forward invariant.*

The second problem considers system (1) with inaccurate state estimation and imperfect actuation. We assume that we only have access to an estimate $\hat{x}_k$ of the true system state x_k with bounded estimation error. Similarly, we assume the real input produced by the imperfect actuator differs from the desired input within a bounded range.

Problem 2: *With inaccurate state estimation and imperfect actuation, design a sampled-data CBF-QP controller shown in* (5) *that renders the set $\mathcal{C}$ forward invariant.*

III. Sampled-Data CBF Condition Based On Interval Analysis

A. Interval Taylor Model of Nonlinear Functions

Solving Problem 1 involves the computation of the range of functions using interval arithmetic. The simplest method is to directly apply interval arithmetic to each term of the function [23], [24]; though fast, this method often results in rather conservative bounds. Instead, the interval Taylor model will be utilized in this paper to obtain tighter bounds of the range of functions.

Definition 1: [Def. 1 in [25]] Let $f : S \subset \mathbb{R}^n \to \mathbb{R}$ be a function that is $(n+1)$ times continuously partially differentiable on an open set containing the domain S. Let x_0 be a point in S and P^n the n-th order Taylor polynomial of f around x_0. Let I be an interval such that $f(x) \in P^n(x - x_0) + I$, $\forall x \in S$. Then the pair (P^n, I) is called an n-th order *Interval Taylor model* of f around x_0 on S.

In general, the reminder interval I will be smaller with a larger value of n.

B. Sampled-data CBF

The main idea of solving Problem 1 is to design a margin term $\phi(x_k, \Delta_k)$ that accounts for the difference between the continuously updated controller and the sampled-data controller, and add it to the CBF condition (6) such that the piecewise-constant controller as shown in (5) can guarantee controlled invariance of set $\mathcal{C}$ in continuous time (i.e. $h(x(t)) \geq 0$ for all $t \geq 0$ whenever $h(x(0)) \geq 0$).

Recall that $x_k = x(t_k)$. Given a CBF h with relative degree r ($r \geq 1$), we call the following inequality

$$L_f^r h(x_k) + L_g L_f^{r-1} h(x_k) u_k + \mathbf{a}^\top \eta(x_k) + \phi(x_k, \Delta_k) \geq 0$$

the *sampled-data CBF* (SDCBF) condition at sampling instance t_k. Define the set

$$K^s_{\text{zcbf}}(x_k, \Delta_k) = \{u \in U \mid L_f^r h(x_k) + L_g L_f^{r-1} h(x_k) u + \mathbf{a}^\top \eta(x_k) + \phi(x_k, \Delta_k) \geq 0\}$$

as the sampled-data admissible input set for the sampled state x_k and the sampling interval Δ_k. Define a function

$$\Delta\xi(x, u, x_k) = \xi(x, u) - \xi(x_k, u)$$

where $\xi(\cdot, u) \triangleq L_f^r h(\cdot) + L_g L_f^{r-1} h(\cdot) u + \mathbf{a}^\top \eta(\cdot)$. Define $z = (x^\top, u^\top)^\top$. For any given sampling interval $\Delta_k > 0$, define the set $\mathcal{Z}_k$ as the Cartesian product of the reachable tube $\mathcal{R}([t_k, t_k + \Delta_k], x_k)$ and the admissible set of the input U, i.e., $\mathcal{Z}_k \triangleq \mathcal{R}([t_k, t_k + \Delta_k], x_k) \times U$. Then we have the following result that solves Problem 1.

Proposition 1: Consider control system (1) and a set $\mathcal{C} \subset \mathbb{R}^n$ defined by (2) for a C^r function $h : \mathbb{R}^n \to \mathbb{R}$ that has a relative degree r. Suppose that $z_k^* = ({x_k^*}^\top, {u_k^*}^\top)^\top$ is a given state-input pair in the set $\mathcal{Z}_k$, i.e, $z_k^* \in \mathcal{Z}_k$, and (P_k^n, I_k) is the n-th Taylor model of $\Delta\xi(x, u, x_k)$ around z_k^*, i.e.,

$$\Delta\xi(x, u, x_k) \in P_k^n(z - z_k^*) + I_k, \ \forall z \in \mathcal{Z}_k. \tag{7}$$

Suppose that $\phi(x_k, \Delta_k)$ is chosen to be

$$\phi(x_k, \Delta_k) = \underline{I}_k + \phi_k^* \tag{8}$$

where $\underline{I}_k$ is the lower bound of I_k, $\phi_k^* = \min_{z \in \mathcal{Z}_k} P_k^n(z - z_k^*)$, and the resulting set $K^s_{\text{zcbf}}(x_k, \Delta_k)$ is non-empty. If $s_k(x(0)) \geq 0$ for $k = 0, 1, ..., r-1$, where s_k are given in (4), then any input $u(t) = u_k, t \in [t_k, t_k + \Delta_k)$ such that $u_k \in K^s_{\text{zcbf}}(x_k, \Delta_k)$ will render $h(x(t)) \geq 0$ for all $t \geq 0$.

Proof: By the definition of $\phi(x_k, \Delta_k)$ and the inclusion relation (7), $\Delta\xi(x, u, x_k) \geq \phi(x_k, \Delta_k)$ holds for any $z \in \mathcal{Z}_k$. For any $t \in [t_k, t_k + \Delta_k)$, since $\Delta\xi(x, u_k, x_k) = \xi(x, u_k) - \xi(x_k, u_k)$, it follows that $L_f^r h(x) + L_g L_f^{r-1} h(x) u_k + \mathbf{a}^\top \eta(x) = L_f^r h(x_k) + L_g L_f^{r-1} h(x_k) u_k + \mathbf{a}^\top \eta(x_k) + \Delta\xi(x, u_k, x_k) \geq L_f^r h(x_k) + L_g L_f^{r-1} h(x_k) u_k + \mathbf{a}^\top \eta(x_k) + \phi(x_k, \Delta_k) \geq 0$, where the last inequality is from the definition of $K^s_{\text{zcbf}}(x_k, \Delta_k)$ and the fact that $u_k \in K^s_{\text{zcbf}}(x_k, \Delta_k)$. Thus, by induction, $L_f^r h(x) + L_g L_f^{r-1} h(x) u + \mathbf{a}^\top \eta(x) \geq 0$ holds for any $t \geq 0$, which implies that h is a CBF for $\mathcal{C}$. Since $u(t)$ is piecewise constant and therefore locally Lipschitz, the conclusion follows by the results of [22]. ■

Note that z_k^* can be any element in the set $\mathcal{Z}_k$. In this paper, we will choose $z_k^* = ({x_k^*}^\top, {u_k^*}^\top)^\top = ({x_k}^\top, {u_c}^\top)^\top$ where u_c is the center of the input admissible set U.

The sampled-data safe controller is obtained by solving the following (SDCBF-QP) only at discrete sampling times:

$$u^*(x_k) = \underset{u \in U}{\text{argmin}} \quad \|u - u_{nom}\|_2 \tag{SDCBF-QP}$$
$$\text{s.t. } L_f^r h(x_k) + L_g L_f^{r-1} h(x_k) u + \mathbf{a}^\top \eta(x_k) + \phi(x_k, \Delta_k) \geq 0$$

where $k \in \mathbb{Z}_{\geq 0}$, and u_{nom} is any given nominal controller.

Next, we show how to compute the term $\phi(x_k, \Delta_k)$ in the SDCBF condition efficiently. For nonlinear systems the exact reachable tube $\mathcal{R}([t_k, t_k + \Delta_k], x_k)$ is generally very challenging to compute, so we will use an over-approximation of $\mathcal{R}([t_k, t_k + \Delta_k], x_k)$, denoted as $\hat{\mathcal{R}}([t_k, t_k + \Delta_k], x_k)$, to compute the Taylor model (7) and the minimization (8). Using the over-approximated reachable tube will result in a smaller $\phi(x_k, \Delta_k)$ which will render the admissible input set $K^s_{\text{zcbf}}(x_k, \Delta_k)$ smaller; however, as long as $K^s_{\text{zcbf}}(x_k, \Delta_k)$ is non-empty for every k, any input $u(t) = u_k, t \in [t_k, t_k + \Delta_k)$ such that $u_k \in K^s_{\text{zcbf}}(x_k, \Delta_k)$ will still guarantee the forward invariance of the set $\mathcal{C}$.

The computation of $\phi(x_k, \Delta_k)$ in Proposition 1 involves two tasks: 1) find $\hat{\mathcal{R}}([t_k, t_k + \Delta_k], x_k)$ and 2) compute $\min_{z \in \mathcal{Z}_k} P_k^n(z - z_k^*)$. In the following, we will discuss how these two tasks can be accomplished efficiently.

1) *Find* $\hat{\mathcal{R}}([t_k, t_k + \Delta_k], x_k)$. To find $\hat{\mathcal{R}}([t_k, t_k + \Delta_k], x_k)$, we will utilize the method in [26], which is based on the linearization of a nonlinear system with interval remainder. Consider a control system given in (1) and recall that $z = (x^\top, u^\top)^\top$. Given a state-input pair $z_k^* = ({x_k^*}^\top, {u_k^*}^\top)^\top = ({x_k}^\top, {u_c}^\top)^\top$, the infinite Taylor series of the i-th state x_i can be overapproximated by its first order Taylor polynomial and its Lagrange remainder as follows:

$$\dot{x}_i \in F_i\left(x_k^*, u_k^*\right) + \left.\frac{\partial F_i(z)}{\partial z}\right|_{z = z_k^*} (z - z_k^*) + L_i([0, 1])$$

where $L_i([0,1]) = \{\frac{1}{2}\left(z - z_k^*\right)^\top \frac{\partial^2 F_i(z)}{\partial z^2}\big|_{z=\zeta} (z - z_k^*) \mid \zeta = z_k^* + \theta(z - z_k^*), \theta \in [0, 1]\} \in \mathbb{IR}$ and z is restricted to a convex set. Therefore, system (1) can be written into the following differential inclusion form:

$$\begin{aligned} \dot{x} &\in F\left(z_k^*\right) + \left.\frac{\partial F(z)}{\partial z}\right|_{z = z_k^*} (z - z_k^*) + L([0, 1]) \\ &= A(x - x_k^*) + B(u - u_k^*) + F\left(x_k^*, u_k^*\right) + L([0, 1]) \end{aligned} \tag{9}$$

where $A = \left(\frac{\partial f(x)}{\partial x} + \frac{\partial g(x)}{\partial x} u\right)\Big|_{x = x_k^*, u = u_k^*} \in \mathbb{R}^{n \times n}$, $B = g(x_k^*) \in \mathbb{R}^{n \times m}$, $L([0,1]) = [L_1([0,1]), \ldots, L_n([0,1])]^\top \in \mathbb{IR}^n$. The over-approximated reachable tube $\hat{\mathcal{R}}([t_k, t_k + \Delta_k], x_k)$ can be obtained by $\hat{\mathcal{R}}([t_k, t_k + \Delta_k], x_k) = R_{lin}(x_k) \oplus R_{err}(x_k)$, where $R_{lin}(x_k)$ is the over-approximated reachable tube of the linearized system shown in (9) with $L = 0$, $R_{err}(x_k)$ is the over-approximated reachable tube of the linearized system resulting from the remainder term L, and $\oplus$ denotes the Minkowski sum. As in [26], we choose zonotopes or intervals as the representation of reachable tubes for computational efficiency. We utilize the same algorithms presented in [26] to compute $R_{lin}(x_k)$ and $R_{err}(x_k)$ for state x_k at each sampling time t_k.

2) *Compute* $\min_{z \in \mathcal{Z}_k} P_k^n(z - z_k^*)$. By the construction of $\hat{\mathcal{R}}([t_k, t_k + \Delta_k], x_k)$ above, the set $\mathcal{Z}_k$ is represented as a zonotope or intervals, which can be readily expressed as a polytope, a more general set representation than zonotope/interval. Specifically, there exists a matrix H_k and a column vector $\mathbf{1}$ whose elements are all 1 with appropriate dimensions such that $\mathcal{Z}_k = \{(x, u) \mid H_k \begin{pmatrix} x \\ u \end{pmatrix} \leq \mathbf{1}\}$.

Since $P_k^n(z-z_k^*)$ is a polynomial with variables x and u, the optimization problem $\min_{z\in\mathcal{Z}_k} P_k^n(z-z_k^*)$ becomes a polynomial optimization problem:

$$\text{(POP)} \quad \phi_k^* = \min_{x,u} \quad P_k^n\left(\begin{pmatrix}x\\u\end{pmatrix}-\begin{pmatrix}x_k^*\\u_k^*\end{pmatrix}\right)$$
$$s.t. \quad H_k\begin{pmatrix}x\\u\end{pmatrix} \le \mathbf{1}$$

A polynomial optimization problem is generally non-convex and known to be NP-hard. It can be solved using non-convex global solvers, such as BMIBNB in YALMIP or using relaxation methods either based on linear programming or semidefinite programming. In this paper we choose to solve (POP) by using Lasserre's linear matrix inequality (LMI) relaxations to obtain a lower bound for ϕ_k^*, the global minimum of (POP) [27]. The relaxed LMIs in Lasserre's hierarchy are convex and can be solved using the interior-point algorithm in polynomial time, and the solutions of the LMIs provide a monotonically nondecreasing sequence of lower bounds for ϕ_k^*. Although the global optimal solution of (POP) can be obtained by increasing the relaxation order, the computational burden increases significantly with larger relaxation order. On the other hand, if $\phi(x_k,\Delta_k)$ is chosen to be $\phi(x_k,\Delta_k)=\underline{I}_k+\underline{\phi}_k$ where $\underline{\phi}_k$ is *any* lower bound of ϕ_k^*, then from the proof of Proposition 1 we know $u_k\in K_{\text{zcbf}}^s(x_k,\Delta_k)$ will still render the set $\mathcal{C}$ forward invariant.

Corollary 1: Suppose that $z_k^*\in\mathcal{Z}_k$, (P_k^n, I_k) is the n-th Taylor model of $\Delta\xi(x,u,x_k)$ around z_k^*, $\phi(x_k,\Delta_k)$ is chosen to be $\phi(x_k,\Delta_k)=\underline{I}_k+\underline{\phi}_k$ where $\underline{I}_k$ is the lower bound of I_k, $\underline{\phi}_k$ is the optimal value of any Lasserre's LMI for (POP), and $K_{\text{zcbf}}^s(x_k,\Delta_k)\neq\emptyset$ for every k. Then any input $u(t)=u_k, t\in[t_k,t_k+\Delta_k)$ such that $u_k\in K_{\text{zcbf}}^s(x_k,\Delta_k)$ will render the set $\mathcal{C}$ forward invariant.

We use SparsePOP to exploit the sparse structure of polynomials when applying Lasserre's hierarchy of LMI relaxations to (POP) [28], and use Mosek to solve the relaxed semidefinite programmings. The computational efficiency of finding $\hat{\mathcal{R}}([t_k,t_k+\Delta_k],x_k)$ and computing $\min_{z\in\mathcal{Z}_k} P_k^n(z-z_k^*)$ will be demonstrated in simulations and experiments in Section V.

Remark 1: In [18], three types of modified CBF conditions for sampled-data systems were proposed. The computation of the CBF conditions there involves non-convex optimization problems that can be solved by nonlinear programming solvers such as FMINCON. However, these solvers are sensitive to the initial conditions, have no guarantee on termination time in general and can only find local optimum values, which make them unsuitable for safety-critical applications. In addition, imperfect state estimation and actuation were not considered in [18]. In [17], a robust backup controller-based CBF controller under state uncertainty is proposed requiring the sampled-data system to be incremental stable. Besides, the CBF condition in [17] involves the nonlinear robust optimization problems which might not be tractable for complex nonlinear dynamics.

Compared with existing results, the proposed framework is applicable to any nonlinear control affine dynamics. In particular, computing $\phi(x_k,\Delta_k)$ is based on convex programs and has several advantages: (i) the related LMIs are convex programs that can be solved efficiently with real-time computation guarantees; (ii) by choosing the order of the Taylor polynomial and the relaxation order of Lasserre's LMI for (POP), we can make a trade-off between the computation's effiency and optimality; (iii) any lower bound of $\phi(x_k,\Delta_k)$ can be used to guarantee the safe set forward invariant as stated in Corollary 1.

IV. Safety under Measurement & Actuation Uncertainties

In practice, the exact state information of a control system is unknown. For sampled-data systems, an estimate of the system state is available at sampling instances, which can be obtained from an observer such as Luenberger or interval observer, or from a Kalman filter. The following assumption provides a measure of the estimation accuracy.

Assumption 1: At any time instance $t_k, k\in\mathbb{Z}_{\ge 0}$, the true state x_k and the estimated state $\hat{x}_k$ satisfy $x_k\in\{\hat{x}_k\}\oplus B_{\epsilon_x}(0)$, where $B_{\epsilon_x}(0)$ is the 2-norm ball at the origin with a radius of $\epsilon_x>0$, i.e., $B_{\epsilon_x}(0)=\{x\in\mathbb{R}^n \mid \|x\|_2\le\epsilon_x\}$.

Assumption 1 means that the measurement uncertainty is bounded by some given parameter ϵ_x, so we have that $x_k\in B_{\epsilon_x}(\hat{x}_k)\triangleq\{x\in\mathbb{R}^n \mid \|x-\hat{x}_k\|_2\le\epsilon_x\}$. Since we only have the estimated state of the system, we will guarantee the forward invariance of the set $\mathcal{C}$ utilizing the enlarged reachable tube $\mathcal{R}([t_k,t_k+\Delta_k],B_{\epsilon_x}(\hat{x}_k))=\{x(t,x_0)\mid x_0\in B_{\epsilon_x}(\hat{x}_k)), t\in[t_k,t_k+\Delta_k], u\in U\}$ and the corresponding set $\hat{\mathcal{Z}}_k$ defined as $\hat{\mathcal{Z}}_k\triangleq\mathcal{R}([t_k,t_k+\Delta_k],B_{\epsilon_x}(\hat{x}_k)))\times U$.

Besides the state estimation uncertainty, the real system might also have an imperfect actuator causing a deviation between the desired input and the real input. To account for the uncertain actuation, we need to bound this deviation and thus guarantee the system safety for the worst-case scenario.

Assumption 2: At any time instance $t_k, k\in\mathbb{Z}_{\ge 0}$, the desired input u_k^d and the real input u_k^r implemented by the system satisfy $u_k^r\in\{u_k^d\}\oplus B_{\epsilon_u}(0)$, where $B_{\epsilon_u}(0)=\{u\in\mathbb{R}^m \mid \|u\|_2\le\epsilon_u\}$.

Suppose that $\hat{z}_k^*=(\hat{x}_k^\top, u_c{}^\top)^\top\in\hat{\mathcal{Z}}_k$ where u_c is the center of the set U. $(\hat{P}_k^n,\hat{I}_k)$ is the n-th Taylor model of $\Delta\xi(x,u,\hat{x}_k)$ around $\hat{z}_k^*$. Let $\phi(\hat{x}_k,\Delta_k)=\hat{\underline{I}}_k+\hat{\phi}_k^*$ where $\hat{\underline{I}}_k$ is the lower bound of $\hat{I}_k$, $\hat{\phi}_k^*=\min_{z\in\hat{\mathcal{Z}}_k}\hat{P}_k^n(z-\hat{z}_k^*)$. Then, we define the new admissible input set as

$$\begin{aligned}\hat{K}_{\text{zcbf}}^s(\hat{x}_k,\Delta_k)=\{&u\in U\ominus B_{\epsilon_u}(0)\mid L_f^r h(\hat{x}_k)\\&+L_gL_f^{r-1}h(\hat{x}_k)u+\mathbf{a}^\top\eta(\hat{x}_k)+\phi(\hat{x}_k,\Delta_k)\ge 0\},\end{aligned}$$

where $\ominus$ is the Pontryagin difference.

The following result provides a solution to Problem 2.

Proposition 2: Consider control system (1) and a set $\mathcal{C}\subset\mathbb{R}^n$ defined by (2) for a C^r function h that is a CBF with a relative degree r such that (3) holds. Suppose $\hat{K}_{\text{zcbf}}^s(\hat{x}_k,\Delta_k)$ is non-empty. If $\min_{x\in B_{\epsilon_x}(\hat{x}_0)} s_k(x)\ge 0$ for $k=0,1,...,r-1$, where s_k are given in (4), then any input $u(t)=u_k^d, t\in[t_k,t_k+\Delta_k)$ such that $u_k^d\in\hat{K}_{\text{zcbf}}^s(\hat{x}_k,\Delta_k)$ will render $h(x(t))\ge 0$ for all $t\ge 0$.

Proof: Since the desired input $u_k^d \in \hat{K}^s_{\text{zcbf}}(\hat{x}_k, \Delta_k)$, according to Assumption 2, the real input $u_k^r \in \hat{K}^s_{\text{zcbf}}(\hat{x}_k, \Delta_k) \oplus B_{\epsilon_u}(0)$ which implies that $u_k^r \in U$ and $L_f^r h(\hat{x}_k) + L_g L_f^{r-1} h(\hat{x}_k) u_k^r + \mathbf{a}^\top \eta(\hat{x}_k) + \phi(\hat{x}_k, \Delta_k) \geq 0$. Using the definition of $\phi(\hat{x}_k, \Delta_k)$, one can get that $L_g L_f^{r-1} h(x) u + L_f^r h(x) + \mathbf{a}^\top \eta(x) \geq 0$ for all $t \geq 0$. Following the same proof as in Proposition 1, it is easy to show the forward invariance of the set $\mathcal{C}$. ■

The sampled-data safe controller with inaccurate state estimation and imperfect actuation is obtained by solving the following robust sampled-data CBF-QP (RSDCBF-QP) only at discrete sampling times:

$$u^d(\hat{x}_k) = \underset{u \in U \ominus B_{\epsilon_u}(0)}{\text{argmin}} \quad \|u - u_{nom}\|_2 \qquad \text{(RSDCBF-QP)}$$

$$\text{s.t. } L_f^r h(\hat{x}_k) + L_g L_f^{r-1} h(\hat{x}_k) u + \mathbf{a}^\top \eta(\hat{x}_k) + \phi(\hat{x}_k, \Delta_k) \geq 0$$

where $k \in \mathbb{Z}_{\geq 0}$, and u_{nom} is any given nominal controller.

V. Simulation & Experiment

In this section, we demonstrate the effectiveness of the proposed method using two examples. For simplicity, we will refer to the sampled-data controller with the naive CBF constraint shown in (6) as "CBF controller" and the safe sampled-data controller by solving (RSDCBF-QP) as "RSDCBF controller".

Example 1: Consider the following dynamics [29]: $\dot{x}_1 = -0.6x_1 - x_2$, $\dot{x}_2 = x_1^3 + x_2 u$. Consider the safe set $\mathcal{C} = \{x \in \mathbb{R}^2 : h(x) \geq 0\}$ where $h(x) = -x_2^2 - x_1 + 1$, which has a relative degree 1. Assume the input is constrained in the set $U = \{u \mid -1 \leq u \leq 1\}$, the periodic sampling time is 0.02 seconds and $\mathbf{a} = 3$. The uncertainty bounds on the estimation error and the actuation error are both 0.1, i.e., $\epsilon_x = \epsilon_u = 0.1$. We choose the nominal controller to be a stabilizing controller based on control Lyapunov functions without considering the safety constraint. The closed-loop system is simulated for 10 seconds starting from the initial state $x_0 = [-2, 1]^\top$. The average computation time (including the computation of $\phi(x_k, \Delta_k)$ and solving (RSDCBF-QP)) at $t = t_0, t_1, ...$ is around 0.018 secs using MATLAB R2020b in a computer with 3.7 GHz CPU and 32 GB memory. Fig. 1 shows system trajectories with the RSDCBF controller and the CBF controller. It can be observed that in the presence of state measurement and actuation uncertainties, the CBF controller can not keep the system safe, while the RSDCBF controller always respects safety.

Example 2: This example presents the experimental results that implements the RSDCBF controller on a Crazyflie Nano Quadcopter. We consider the following linearized 6-dimension quadcopter model:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0_{3\times 3} \\ I_{3\times 3} \end{bmatrix} \mathbf{u}. \tag{10}$$

where $\mathbf{x} = [x \;\; y \;\; z \;\; \dot{x} \;\; \dot{y} \;\; \dot{z}]^T$ is the state and $\mathbf{u} = [\ddot{x} \;\; \ddot{y} \;\; \ddot{z}]^\top$ is the virtual input. The model shown in (10) is usually referred to as the double-integrator quadcopter model and is widely used in quadcopter simulations [30]. Define the safe set $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^6 : \mathbf{h}(\mathbf{x}) \geq \mathbf{0}\}$ with $\mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x}),\ h_2(\mathbf{x}),\ h_3(\mathbf{x}),\ h_4(\mathbf{x}),\ h_5(\mathbf{x}),\ h_6(\mathbf{x})]^\top = [0.5 - x,\ x + 0.5,\ 0.5 - y,\ y + 0.5,\ 0.6 - z,\ z]^\top$. The SDCBF condition for $i = 1, ..., 6$ is given by $L_f^2 h_i(\mathbf{x}) + L_g L_f h_i(\mathbf{x})\mathbf{u} + \mathbf{a}^\top \eta_i(\mathbf{x}) + \phi_i(\mathbf{x}, \Delta t) \geq 0$, where $\eta_i(\mathbf{x}) = [L_f h_i(\mathbf{x}), h_i(\mathbf{x})]^\top$ and $\mathbf{a} = [6, 8]^\top$. We use a LQR controller as the nominal controller to track a given reference trajectory. We choose $\epsilon_x = 0.02$ and $\epsilon_u = 0.01$.

The quadcopter flies for about 20 seconds to follow the reference trajectory starting from the origin. We implement both CBF and RSDCBF controllers in the quadcopter experiments, with the periodic sampling time as 0.02 seconds. Fig. 2 illustrates the reference trajectory and the quadcopter trajectories with two types of CBF-based controllers. Although most of the trajectory using the CBF controller is inside the constraining box (the safe set $\mathcal{C}$), $\mathbf{h}(\mathbf{x}) \geq \mathbf{0}$ is violated at some extreme points in Fig. 3. In contrast, the trajectory using the RSDCBF controller remains in the constraining box for all time. The experimental results show that the RSDCBF controller can guarantee safety under measurement and actuation uncertainties, which is necessary for the quadcopter and other safety-critical robotic applications. The average computation time is 0.005 seconds at each sampling time.

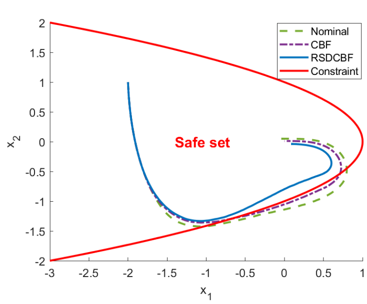

Fig. 1: Simulation result for Example 1. System trajectories with three controllers are shown: (i) nominal controller (green), which is obtained from control Lyapunov function conditions; (ii) CBF controller (purple), which is a sampled-data controller with the naive CBF constraint shown in (6); (iii) RSDCBF controller (blue), which is obtained by solving (RSDCBF-QP). The boundary of safe set $\mathcal{C}$ is shown in red.

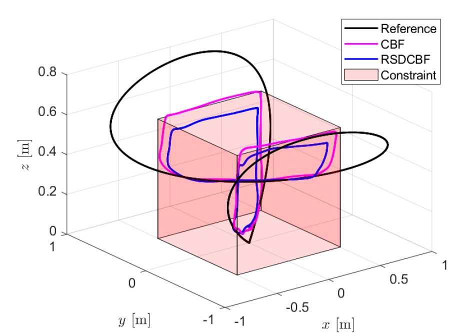

Fig. 2: Experimental results for Example 2. Trajectories of Crazyflie with the CBF controller (pink), the RSDCBF controller (blue), and the reference trajectory are shown.

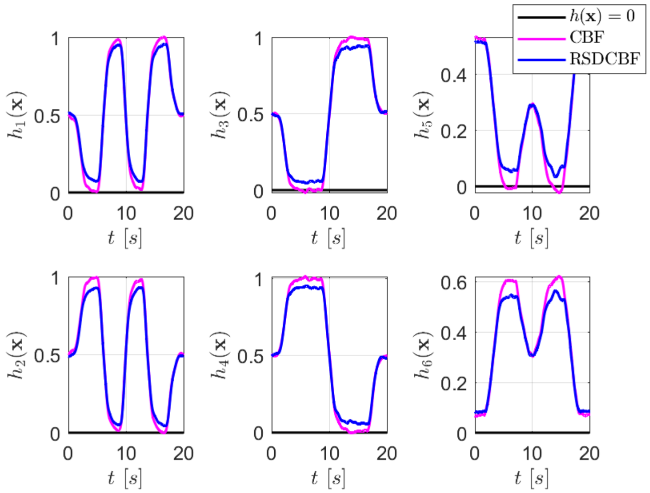

Fig. 3: Evolution of CBF $\mathbf{h}(\mathbf{x})$ in Example 2. The RSDCBF-QP controller respects all safety constraints while the CBF controller induces safety violations.

VI. Conclusion

In this paper, we proposed a framework that can guarantee the safe control of sampled-data systems with measurement and actuation uncertainties. Comparing with the traditional CBF condition, the proposed SDCBF condition includes an additional term $\phi(x_k, \Delta_k)$ which can be efficiently solved by computing the lower bound of a Taylor polynomial using reachable tube approximation and polynomial optimization techniques. We proved that the SDCBF-QP controller can guarantee the safety constraint in continuous time for sampled-data systems with perfect information and the RSDCBF-QP controller can ensure safety with inaccurate state estimation and imperfect actuation. The performance of the proposed method was demonstrated via simulations and hardware experiments on the quadcopter. Future work includes approximating the model uncertainties using online measurements and applying the proposed framework to more realistic robotic applications such as navigating or object tracking in unknown environment.

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