Observer-based Control Barrier Functions for Safety Critical Systems

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Abstract— **This paper considers the safety-critical control design problem with output measurements. An observer-based safety control framework that integrates the estimation error quantified observer and the control barrier function (CBF) approach is proposed. The function approximation technique is employed to approximate the uncertainties introduced by the state estimation error, and an adaptive CBF approach is proposed to design the safe controller which is obtained by solving a convex quadratic program (QP). Theoretical results for CBFs with a relative degree 1 and a higher relative degree are given individually. The effectiveness of the proposed control approach is demonstrated by two numerical examples.**

I. Introduction

The safety of autonomous systems has drawn increasing attention in the past decades with various control techniques been developed [1]–[3]. Control barrier function (CBF) has become a powerful tool for achieving safety in the form of set invariance [4], [5], and has been applied to many scenarios including adaptive cruise control [6], biped robots [7], and UAVs [8]. However, most existing results using CBFs rely on accurate state information, which is hard to obtain in practice. Different methods have been proposed to address this issue [9]–[13]. In [9], a function mapping from outputs to states is learned via supervised learning techniques, and the controller is designed under the assumption that for any given output value, all possible state estimation error is bounded by a known constant. Khalaf *et al.* [10] proposed a controller synthesis approach involving feedback from pixels, which does not require feature extraction, object detection, or state estimation. Poonawala *et al.* [11] developed a method that trains classifiers for sensor-based control problems, bypassing the state estimation step. Takano and Yamakita [12] proposed a QP-based controller with an unscented Kalman filter which is capable of attenuating the effects of state disturbances and measurement noises. Garg and Panagou [13] employed robust CBF and fixed-time control Lyapunov functions to guarantee stability and safety of the system in the presence of additive disturbance and state estimation error. In spite of these interesting results, assumptions in these works are difficult to satisfy that limit their applicability; for example, in [13], the derivative of the CBF is assumed to be bounded, and the uncertainties introduced by state estimation error are not taken into account.

This paper considers the safety-critical control design problem with observers. The main contribution of this work lies in a novel observer-based CBF framework that integrates the estimation error quantified (EEQ) observer and the CBF approach, which generates provably safe controllers under mild conditions. A new safe set is defined to guarantee that the true state is in the original safe set provided the state estimation is in the new safe set. The residual terms introduced by the state estimation error are approximated by the function approximation technique (FAT) via a given set of basis functions with unknown weights. An adaptive CBF method is proposed to guarantee the safety of the controlled system, where the unknown weights are estimated by adaptive laws. The EEQ observer considered in this work can be not only the traditional asymptotic observer but also interval observers [14] and neural-network-based observers [15].

The rest of this paper is organized as follows. Section II reviews some preliminaries about FAT, CBF and estimation error quantified observers, and presents the problem statement. Section III provides the main result of this paper for CBFs with a relative degree 1 and a higher relative degree, respectively. The effectiveness of the proposed control design method is demonstrated by simulations in Section IV. Finally, conclusions are drawn in Section V.

II. Preliminaries & Problem Statement

A. Function Approximation Technique

FAT is an effective tool for approximating time-varying unknown nonlinear functions [16]. Its basic idea is to express an unknown nonlinear function as the combination of a set of given basis functions. There are a lot of examples of FAT, such as the generalized Fourier series [16] and neural networks [17]. In this paper, the basis functions are selected as trigonometric functions (the basis functions of Fourier series), which have been widely used in previous works [18]–[20]. There are other options for basis functions, including Bernstein polynomials, Legendre polynomials, and Chebyshev polynomials.

An arbitrary square integrable function $f(t):\mathbb{R}\to\mathbb{R}$ can be approximated by a generalized Fourier series in the interval $[t_1,t_2]$ as $f(t)=\sum_{i=1}^{N}\theta_i\varphi_i(t)+\epsilon_N(t)$ [20], where N is a given integer, θ_i is the corresponding coefficient, $\epsilon_N(t)$ is the truncation error satisfying $\lim_{N\to\infty}\int_{t_1}^{t_2}|\epsilon_N(t)|^2\mathrm{d}t=0$, and $\varphi_i(t), i=1,2,\cdots,N$ are trigonometric basis functions. Since $\epsilon_N(t)$ vanishes as $N\to\infty$, $f(t)$ can be expressed as $f(t)=\sum_{i=1}^{\infty}\theta_i\varphi_i(t)$. As $f(t)$ is an unknown function, θ_i cannot be directly calculated from the integral of $f(t)$. Hence, the majority of FAT-based controllers employ adaptive control techniques to estimate θ_i online, such that the estimation of $f(t)$ can be expressed as $\hat{f}(t)=\sum_{i=1}^{N}\hat{\theta}_i\varphi_i(t)$, where $\hat{\theta}_i$ is the estimation of θ_i and governed by corresponding adaptive laws [18]. For a vector function $f(t):\mathbb{R}\to\mathbb{R}^n$,

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the approximation above holds for a set of vector parameters $\theta_i, \hat{\theta}_i \in \mathbb{R}^n$.

B. Control Barrier Function

Consider the following control affine system with output measurement:

$$\dot{x} = f(x) + g(x)u, \tag{1}$$
$$y = l(x), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in U \subset \mathbb{R}^m$ is the control input, $l : \mathbb{R}^n \to \mathbb{R}^k$ is the output measurement, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipchitz continuous functions. A set $\mathcal{S}$ is called forward controlled invariant with respect to system (1) if for every $x_0 \in \mathcal{S}$, there exists a control signal $u(t)$ such that $x(t; t_0, x_0) \in \mathcal{S}$ for all $t \geq t_0$, where $x(t; t_0, x_0)$ denotes the solution of (1) at time t with initial condition $x_0 \in \mathbb{R}^n$ at time t_0. To simplify the discussion, we will use the same definition as above for the controlled invariance of time-varying systems, which is slightly different from the definition given in [21].

Consider control system (1) and a set $\mathcal{C} \subset \mathbb{R}^n$ defined by

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\} \tag{3}$$

for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ that has a relative degree 1. The function h is called a (zeroing) CBF if there exists a constant $\gamma > 0$ such that

$$\sup_{u\in U} [L_f h(x) + L_g h(x)u + \gamma h(x)] \geq 0 \tag{4}$$

where $L_f h(x) = \frac{\partial h}{\partial x} f(x)$ and $L_g h(x) = \frac{\partial h}{\partial x} g(x)$ are Lie derivatives [5]. Given a CBF h, the set of all control values that satisfy (4) for all $x \in \mathbb{R}^n$ is defined as: $K_{\text{zcbf}}(x) = \{u \in U : L_f h(x) + L_g h(x)u + \gamma h(x) \geq 0\}$. It was proven in [5] that any Lipschitz continuous controller $u(x) \in K_{\text{zcbf}}(x)$ for every $x \in \mathbb{R}^n$ will guarantee the forward invariance of $\mathcal{C}$. The provably safe control law is obtained by solving an online quadratic program (QP) that includes the control barrier condition as its constraint.

A C^r function $h(x) : \mathbb{R}^n \to \mathbb{R}$ with a relative degree r where $r \geq 2$ is called a (zeroing) CBF if there exists a column vector $\mathbf{a} \in \mathbb{R}^r$ such that $\forall x \in \mathbb{R}^n$,

$$\sup_{u\in U} [L_g L_f^{r-1} h(x)u + L_f^r h(x) + \mathbf{a}'\eta(x)] \geq 0 \tag{5}$$

where $\eta(x) = [L_f^{r-1}h, L_f^{r-2}h, ..., h]^\top \in \mathbb{R}^r$, and $\mathbf{a} = [a_1, ..., a_r]^\top \in \mathbb{R}^r$ is chosen such that the roots of $p_0^r(\lambda) = \lambda^r + a_1\lambda^{r-1} + ... + a_{r-1}\lambda + a_r$ are all negative reals $-\lambda_1, ..., -\lambda_r < 0$. Define functions $s_k(x(t))$ for $k = 0, 1, ..., r$ as follows:

$$s_0(x(t)) = h(x(t)),\ s_k(x(t)) = (\frac{\mathrm{d}}{\mathrm{d}t} + \lambda_k)s_{k-1}. \tag{6}$$

It was shown in [22] that if $s_k(x(0)) > 0$ for $k = 0, 1, ..., r-1$, then any controller $u(x) \in \{u \in U : L_g L_f^{r-1} h_0(x)u + L_f^r h_0(x) + \mathbf{a}'\eta(x)] \geq 0\}$ that is locally Lipschitz will guarantee the forward invariance of $\mathcal{C}$. The time-varying CBF with a general relative degree and its safety guarantee for a time-varying system were discussed in [23].

C. Estimation Error Quantified Observer

An observer for system (1)-(2) is given as:

$$\dot{\hat{x}} = v(\hat{x}, y, u) \tag{7}$$

where $\hat{x}$ is the estimated state, u is the input, and y is the output. Define the state estimation error as $e = \hat{x} - x$. Then the *error dynamics* is given as

$$\dot{e} = \bar{v}(e, \hat{x}, y, u) \tag{8}$$

where $\bar{v}(e, \hat{x}, y, u) = v(\hat{x}, y, u) - f(\hat{x} - e) - g(\hat{x} - e)u$.

In this paper, we consider the *estimation error quantified (EEQ) observer* that is a generalization of the traditional asymptotic observer which requires the state estimation error to converge to zero. The definition of the EEQ observer is introduced below.

Definition 1: An observer is called an EEQ observer for system (1)-(2) if it provides a state estimation $\hat{x}(t)$ such that

$$\|\hat{x}(t) - x(t)\| \leq M(t, x_0, \hat{x}_0), \quad \forall t \geq 0, \tag{9}$$

where $M(t, x_0, \hat{x}_0) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a known time-varying function whose values are non-negative.

To simply the notation, we will use $M(t)$ for $M(t, x_0, \hat{x}_0)$ in the following. The EEQ observer subsumes many types of common observers.

1) Interval Observers: The interval observer is an observer that provides an estimation interval for the true states by using the input-output measurement [14]. Specifically, an interval observer has two dynamic systems that provide the upper bound $\bar{x}(t)$ and lower bound $\underline{x}(t)$ of the true state $x(t)$, respectively. If its state estimation is selected as $\hat{x}(t) = \frac{\bar{x}(t)+\underline{x}(t)}{2}$, then an interval observer is an EEQ observer with $M(t)$ shown in (9) chosen as $M(t) = \frac{1}{2}\|\bar{x}(t) - \underline{x}(t)\|$.

2) Exponentially Stable Observers: According to [24, p. 150], an (global) exponentially stable observer requires that the equilibrium point $e = 0$ of the error system shown in (8) is exponentially stable, i.e., there exist positive constants k, and λ such that $\|e(t)\| \leq k\|e(0)\|\exp(-\lambda t)$. An exponentially stable observer is an EEQ observer with $M(t)$ shown in (9) chosen as $M(t) = D\exp(-\lambda t)$, where D is a positive constant satisfying $D \geq k\|e(0)\|$. Note that any exponentially stable observer can be employed in the proposed control scheme.

3) Neural-Network-Based Observer: Deep neural networks can be employed to design observers because of its universal approximation property [15], [25]. For example, in [15], a neural network is trained to approximate the function ψ, which recovers x from y by $x = \psi(y)$, such that the state estimation is given by a trained function as $\hat{x} = r(y, \alpha)$, where α is the training parameter. There are many techniques that can be used to train the neural networks, such as stochastic gradient decent and Adam algorithm. Nevertheless, since the neural network is trained on the training set E_S, its approximation accuracy on the complete dataset E is not always guaranteed even if the approximation error is small enough on E_S. Marchi *et al.* [15] pointed out that if E_S is a δ-cover of E with respect to a certain partial order on $\mathbb{R}^n$, then any continuous function ψ can be approximated by a function $r = \phi + A$, where ϕ is a monotone function and A is a linear map, with the generalization error

$$\|\psi - r\|_{L^\infty(E)} \le 3\|\psi - r\|_{L^\infty(E_S)} + 2\omega_\psi(\delta) + 2\|A\|_\infty \delta, \quad (10)$$

where ω_ψ is a modulus of continuity of ψ on E and $\|A\|_\infty$ denotes the operator ∞-norm of the map A. Note that $\delta = 4\eta$ if E_S is an η-cover of S. Thus, the neural-network-based observer r designed in [15] is an EEQ observer with $M(t)$ shown in (9) chosen as $M(t) = \beta$, where β is the constant on the right hand side of (10).

D. Problem Statement

This paper will consider the CBF-based safety control design problem with an EEQ observer in the loop. Specifically, the problem that will be studied is given as follows.

Problem 1: *Given system* (1)-(2) *and its EEQ observer* (9)*, design a feedback controller $u(\hat{x}) : \mathbb{R}^n \to \mathbb{R}^m$ such that the trajectory of the closed-loop system will stay inside the safe set $\mathcal{C}$ defined in* (3)*, i.e., $h(x(t)) \ge 0$ for all $t \ge 0$.*

III. Main Results

In this section, we reconstruct system (1) whose state x cannot be known exactly into a model of $\hat{x}$, which is the state of the EEQ observer, by using FAT introduced in Section II-A. Based on that, we develop an adaptive CBF method to design the safe controller for CBFs with a relative degree 1 and a higher relative degree individually.

Recalling the definition of the state estimation error e, system (1) can be rewritten as

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + \Lambda(x, \dot{e}, \hat{x}, u) \quad (11)$$

where

$$\Lambda(x, \dot{e}, \hat{x}, u) = \dot{e} + \delta f + \delta g u \in \mathbb{R}^n$$

with $\delta f = f(x) - f(\hat{x}) \in \mathbb{R}^n$ and $\delta g = g(x) - g(\hat{x}) \in \mathbb{R}^{n \times m}$. Since x, $\dot{e}$, $\hat{x}$ are solutions of the closed-loop system composed of systems (1)-(2) and its EEQ observer (9), they are variables with respect to time t. Thus, $\Lambda(x, \dot{e}, \hat{x}, u)$ can also be seen as a function of t, which can be approximated by using trigonometric functions as the basis function as follows:

$$\Lambda(x, \dot{e}, \hat{x}, u) = \sum_{i=1}^{N} \theta_i \varphi_i(t) + \epsilon_\Lambda \quad (12)$$

where $\varphi_i \in \mathbb{R}$ represents the set of trigonometric scalar functions, N is a positive integer, $\epsilon_\Lambda \in \mathbb{R}^n$ denotes the truncation error, and $\theta_i \in \mathbb{R}^n$ is a vector of parameters that are unknown constants. Substituting (12) into (11) yields

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + \sum_{i=1}^{N} \theta_i \varphi_i(t) + \epsilon_\Lambda. \quad (13)$$

It can be seen that the reconstructed system (13) is a model of $\hat{x}$ containing unknown parameters. We will solve Problem 1 by considering (13) and using an adaptive control design method. The control input u will be designed to render the set $\mathcal{C}$ safe with regard to system (13) in the presence of unknown parameters θ_i. Two assumptions regarding ϵ_Λ and θ_i are proposed as follows.

Assumption 1: [26] There exists a positive constant $E > 0$ such that $\|\epsilon_\Lambda\| \le E$.

Assumption 2: [27] There exist constants $\bar{\theta}_i$ for $i = 1, ..., N$, such that the unknown parameter θ_i in (13) is bounded by $\bar{\theta}_i$, i.e. $\|\theta_i\| \le \bar{\theta}_i$.

Remark 1: According to [28], if a continuous function defined on $[0, T]$ with respect to t satisfies the Dirichlet condition, it can expanded as in (12), where T is the fundamental period of Λ. Therefore, we need to use the FAT repeatedly or select sufficiently large T. □

Remark 2: Given an arbitrary constant $E > 0$, one can choose N large enough to make the truncation error $\|\epsilon_\Lambda\|$ smaller than E. Although a better approximation accuracy can be achieved with a smaller E, the corresponding computational burden may grow up rapidly with the increase of N. Moreover, if N is too large, Gibbs phenomenon may appear. Our past experience indicates that in most cases, $N \le 5$ is sufficient to guarantee a good approximation accuracy [29]. Furthermore, $\bar{\theta}_i$ should be treated as tuning parameters. In practice, the optimal uncertainty bounds cannot be known exactly and are usually tuned via trial-and-error [27]. □

A. Safe Control Design for CBF with Relative Degree 1

In this subsection, a feedback controller will be designed for system (13) to solve Problem 1 where the CBF h is assumed to have a relative degree 1.

Note that the state variable of system (13) is $\hat{x}$ instead of x, while $h(x)$ is a function of x. Hence, we try to construct a new function $\bar{h}$ guaranteeing $h(x) \ge 0$ provided $\bar{h}(\hat{x}) \ge 0$. Assume $h(x)$ is a global Lipschitz function. Thus, there exists a constant $L > 0$ as the Lipschitz constant of h, such that for all $x, \hat{x}$,

$$|h(x) - h(\hat{x})| \le L\|x - \hat{x}\| \quad (14)$$

where $x, \hat{x}$ are the true and estimated states of system (1), respectively, which implies that

$$h(x) \ge h(\hat{x}) - L\|x - \hat{x}\| \ge h(\hat{x}) - LM(t) \quad (15)$$

where the last inequality is from (9). Define a time-varying function $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as

$$h_0(x, t) = h(x) - LM(t). \quad (16)$$

From (15) and the fact that $M(t) \ge 0$ for $t \ge 0$, it is clear that $h_0(\hat{x}, t) \ge 0$ implies $h(x) \ge 0$ for any $t \ge 0$. Therefore, if a controller can be designed such that $h_0(\hat{x}, t) \ge 0$ for $t \ge 0$, then the forward invariance of $\mathcal{C}$ will be guaranteed.

Remark 3: Assume that the set of initial conditions $x_0, \hat{x}_0$ renders $M(t) \le M_0$ for $t \ge 0$, where $M_0 > 0$ is a fixed positive constant. Define $\mathcal{C}^{M_0} \triangleq \mathcal{C} \oplus B_{M_0}(0)$ where $\oplus$ is the Minkowski sum and $B_{M_0}(0)$ is the 2-norm ball at the origin with a radius of M_0. Then the global Lipschitz requirement on h shown in (14) can be relaxed to the condition that h has a Lipschitz constant L on $\mathcal{C}^{M_0}$.

Define a new function $h_\epsilon : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as

$$h_\epsilon(\hat{x}, t) = h_0(\hat{x}, t) - \epsilon \quad (17)$$

where h_0 is given in (16), and $\epsilon > 0$ is a positive constant that will be determined later. The following theorem is the main result of this subsection.

Theorem 1: Consider system (13) with unknown parameters θ_i and a set $\mathcal{C}$ defined in (3) for a continuously differentiable function h. Suppose that h has a relative degree 1 and satisfies (14) for some $L > 0$. Assume that Assumption 1 and 2 hold. Suppose that the parameter estimation $\hat{\theta}_i$ is governed by the following adaptive law:

$$\dot{\hat{\theta}}_i = -\frac{\bar{\theta}_i^2}{2\epsilon}\frac{\partial h_\epsilon}{\partial \hat{x}}\varphi_i - \mu\hat{\theta}_i, \tag{18}$$

where $\mu > 0$ a positive constant, and h_ϵ is given in (17). If ϵ is chosen such that

$$0 < \epsilon \leq \frac{h_0(\hat{x}(0), 0)}{\left(N + \sum_{i=1}^N \left(\frac{2\|\hat{\theta}_i(0)\|}{\bar{\theta}_i} + \frac{\|\hat{\theta}_i(0)\|^2}{\bar{\theta}_i^2}\right)\right)}, \tag{19}$$

then any Lipschitz continuous controller $u(x) \in K_{BF}(\hat{x}, \hat{\theta}, t) \triangleq \{u \mid \psi_0 + \psi_1 u \geq 0\}$ where

$$\psi_0 = L_{\tilde{f}}h_\epsilon - \mu N\epsilon + \frac{\partial h_\epsilon}{\partial t} - \left\|\frac{\partial h_\epsilon}{\partial \hat{x}}\right\|E + \mu h_\epsilon, \tag{20a}$$

$$\psi_1 = L_g h_\epsilon, \tag{20b}$$

and $\tilde{f} = f + \sum_{i=1}^N \hat{\theta}_i\varphi_i$ will guarantee the safety of $\mathcal{C}$ in regard to system (13).

Proof: Define a composite CBF candidate as

$$\bar{h}(\hat{x}, \hat{\theta}, t) = h_0(\hat{x}, t) - \sum_{i=1}^N \frac{\epsilon}{\bar{\theta}_i^2}\tilde{\theta}_i^\top\tilde{\theta}_i, \tag{21}$$

where $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ represents the parameter estimation error and $\hat{\theta} = [\hat{\theta}_1, \cdots, \hat{\theta}_N]^T$. The time derivative of $\bar{h}$ is

$$\begin{aligned}
&\dot{\bar{h}}(\hat{x}, \hat{\theta}, t) \\
&= \frac{\partial h_0}{\partial \hat{x}}^\top(f(\hat{x}) + g(\hat{x})u + \sum_{i=1}^N \theta_i\varphi_i + \epsilon_\Lambda) + \frac{\partial h_0}{\partial t} + \sum_{i=1}^N \frac{2\epsilon}{\bar{\theta}_i^2}\tilde{\theta}_i^\top\dot{\hat{\theta}}_i \\
&= \frac{\partial h_\epsilon}{\partial \hat{x}}^\top(f(\hat{x}) + g(\hat{x})u + \sum_{i=1}^N \hat{\theta}_i\varphi_i + \epsilon_\Lambda) + \frac{\partial h_\epsilon}{\partial t} \\
&\quad + \sum_{i=1}^N \tilde{\theta}_i^\top\left(\frac{\partial h_\epsilon}{\partial \hat{x}}\varphi_i + \frac{2\epsilon}{\bar{\theta}_i^2}\dot{\hat{\theta}}_i\right).
\end{aligned} \tag{22}$$

Substituting (18) into (22) gives

$$\dot{\bar{h}}(\hat{x}, \hat{\theta}, t) = L_g h_\epsilon u + L_{\tilde{f}}h_\epsilon + \frac{\partial h_\epsilon}{\partial \hat{x}}^\top\epsilon_\Lambda + \frac{\partial h_\epsilon}{\partial t} - \sum_{i=1}^N \frac{2\mu\epsilon}{\bar{\theta}_i^2}\tilde{\theta}_i^\top\hat{\theta}_i.$$

It can be seen that $u \in K_{BF}(\hat{x}, \hat{\theta}, t)$ indicates

$$\begin{aligned}
\dot{\bar{h}}(\hat{x}, \hat{\theta}, t) &\geq \frac{\partial h_\epsilon}{\partial \hat{x}}^\top\epsilon_\Lambda + \left\|\frac{\partial h_\epsilon}{\partial \hat{x}}\right\|E - \mu h_\epsilon - \sum_{i=1}^N \frac{2\mu\epsilon}{\bar{\theta}_i^2}\tilde{\theta}_i^\top\hat{\theta}_i + \mu N\epsilon \\
&\geq -\mu h_0(\hat{x}, t) + \mu(N+1)\epsilon - \sum_{i=1}^N \frac{2\mu\epsilon}{\bar{\theta}_i^2}\tilde{\theta}_i^\top\hat{\theta}_i
\end{aligned} \tag{23}$$

where the first inequality is from $u \in K_{BF}(\hat{x}, \hat{\theta}, t)$, and the second inequality is derived from the assumption $\|\epsilon_\Lambda\| \leq E$, which is stated in Assumption 1. Since $\tilde{\theta}_i^\top\hat{\theta}_i = \tilde{\theta}_i^\top(\theta_i - \tilde{\theta}_i) \leq \frac{\theta_i^\top\theta_i}{2} - \frac{\tilde{\theta}_i^\top\tilde{\theta}_i}{2}$, from (23) one gets

$$\dot{\bar{h}}(\hat{x}, \hat{\theta}, t) \geq -\mu\bar{h}(\hat{x}, \hat{\theta}, t) + \mu(N+1)\epsilon - \sum_{i=1}^N \frac{\mu\epsilon}{\bar{\theta}_i^2}\theta_i^\top\theta_i. \tag{24}$$

As $\|\theta_i\| \leq \bar{\theta}_i$, one obtains $\dot{\bar{h}}(\hat{x}, \hat{\theta}, t) \geq -\mu\bar{h}(\hat{x}, \hat{\theta}, t)$. By the comparison lemma [24, Page 103], we get

$$\bar{h}(t) \geq \bar{h}(\hat{x}(0), \hat{\theta}(0), 0)e^{-\mu t}. \tag{25}$$

Since

$$\bar{h}(\hat{x}, \hat{\theta}, t) \geq h_0(\hat{x}, t) - \sum_{i=1}^N \frac{\epsilon}{\bar{\theta}_i^2}(\|\hat{\theta}_i\|^2 + 2\|\hat{\theta}_i\|\bar{\theta}_i + \bar{\theta}_i^2), \tag{26}$$

when $t = 0$, substituting (19) into (26) yields

$$\begin{aligned}
\bar{h}(\hat{x}(0), \hat{\theta}(0), 0) &\geq \epsilon\left(N + \sum_{i=1}^N \left(\frac{2\|\hat{\theta}_i(0)\|}{\bar{\theta}_i} + \frac{\|\hat{\theta}_i(0)\|^2}{\bar{\theta}_i^2}\right)\right) \\
&\quad - \sum_{i=1}^N \frac{\epsilon}{\bar{\theta}_i^2}(\|\hat{\theta}_i(0)\|^2 + 2\|\hat{\theta}_i(0)\|\bar{\theta}_i + \bar{\theta}_i^2).
\end{aligned} \tag{27}$$

Obviously, (27) indicates $\bar{h}(\hat{x}(0), \hat{\theta}(0), 0) \geq 0$. Therefore, according to (25), one gets $\bar{h}(t) \geq 0$ for $t \geq 0$. From (21), it can be seen that $\bar{h}(t) \geq 0$ indicates $h_0(\hat{x}, t) \geq 0$. Moreover, $h_0(\hat{x}, t) \geq 0$ implies $h(x) \geq 0$. Hence, the set $\mathcal{C}$ is safe with regard to system (13). ■

By Theorem 1, the safe controller for solving Problem 1 is obtained by the following CBF-QP:

$$\begin{aligned}
\min_u \quad & \|u - u_d\|^2 \\
\text{s.t.} \quad & \psi_0 + \psi_1 u \geq 0, \\
& \hat{\theta}_i \textit{ is given by } (18),
\end{aligned} \tag{28}$$

where ψ_0, ψ_1 are given in (20), and u_d is any given nominal control that might be unsafe.

B. Safe Control Design for CBF with High Relative Degree

In this subsection, we will consider solving Problem 1 for a CBF h that is assumed to have a relative degree $r \geq 2$. Recall the definition of functions $s_k(x(t))$ shown in (6). Assume that $s_k(x(t))$ is globally Lipschitz and has Lipschitz constant $L_k > 0$ for $k = 0, 1, \cdots, r-1$. Note that this requirement can be relaxed as metioned in Remark 3. Define a family of sets:

$$\mathcal{C}_k = \{x \in \mathbb{R}^n : s_k(x) \geq 0\}, \ k = 0, 1, \cdots, r-1, \tag{29}$$

and a family of functions:

$$s_k^M(x, t) = s_k(x) - L_k M(t), \ k = 0, 1, \cdots, r-1. \tag{30}$$

Theorem 2: Consider the system (13) with unknown parameters θ_i and a set $\mathcal{C}$ defined in (3) for a C^r function h that has a relative degree r. Assume that $s_k(x)$ has Lipschitz constant L_k for $k = 0, 1, ..., r$. Suppose that Assumptions 1 and 2 hold and the parameter estimation $\hat{\theta}_i$ is governed by the following adaptive law:

$$\dot{\hat{\theta}}_i = -\frac{\bar{\theta}_i^2}{2\epsilon}\frac{\partial s_\epsilon}{\partial \hat{x}}\varphi_i - \mu\hat{\theta}_i, \tag{31}$$

where $\mu > 0$ a positive constant and $s_\epsilon(\hat{x}, t) = s_{r-1}^M(\hat{x}, t) - \epsilon$ with s_k^M defined in (30). If there exists ϵ such that

$$0 < \epsilon \leq \frac{\min\{s_0^M(\hat{x}(0), 0), s_1^M(\hat{x}(0), 0), \cdots, s_{r-1}^M(\hat{x}(0), 0)\}}{\left(N + \sum_{i=1}^N \left(\frac{2\|\hat{\theta}_i(0)\|}{\bar{\theta}_i} + \frac{\|\hat{\theta}_i(0)\|^2}{\bar{\theta}_i^2}\right)\right)}, \tag{32}$$

then any Lipschitz continuous controller $u(x) \in K_{BF,r}(\hat{x}, \hat{\theta}, t) \triangleq \{u \mid \psi_0^r + \psi_1^r u \geq 0\}$ where

$$\psi_0^r = L_{\tilde{f}} s_\epsilon - \mu N \epsilon + \frac{\partial s_\epsilon}{\partial t} - \left\| \frac{\partial s_\epsilon}{\partial \hat{x}} \right\| E + \mu s_\epsilon, \tag{33a}$$

$$\psi_1^r = L_g s_\epsilon, \tag{33b}$$

and $\tilde{f} = f + \sum_{i=1}^N \hat{\theta}_i \varphi_i$ will guarantee the invariance of $\mathcal{C}$.

Proof: Similar to (15) and (16), it can be seen that $s_k^M(\hat{x}, t) \geq 0$ indicates $s_k(x) \geq 0$. From (32), one gets $s_k^M(\hat{x}(0), 0) \geq 0$, such that $s_k(x(0)) \geq 0$. Following the same procedure as shown in the proof of Theorem 1, one gets $s_{r-1}(x) \geq 0$. As $s_k(x(0)) \geq 0$ holds for each $k = 0, 1, \cdots, r-1$, according to [22, Theorem 1], one has that $\mathcal{C}_k$ is forward invariant. Since $\mathcal{C} = \mathcal{C}_0$, the conclusion follows immediately. ■

By Theorem 2, the safe controller for solving Problem 1 where the CBF has a relative degree $r \geq 2$ is obtained by the following CBF-QP:

$$\begin{aligned} \min_u \quad & \|u - u_d\|^2 \\ \text{s.t.} \quad & \psi_0^r + \psi_1^r u \geq 0, \\ & \hat{\theta}_i \textit{ is given by } (31), \end{aligned} \tag{34}$$

where ψ_0^r, ψ_1^r are given in (33), and u_d is any given nominal control that might be unsafe.

IV. Simulation

In this section, we use two numerical examples to illustrate the effectiveness of the proposed control scheme.

Example 1: Consider a linear system

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{B} u,$$

$$y = \underbrace{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}_{C} x.$$

A Luenberger observer is designed as $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ where $L = \begin{bmatrix} -2.23029 & 0.190287 & 0.232326 \end{bmatrix}^\top$ such that $A - LC$ is Hurwitz. Consider the safe set $\mathcal{C} \triangleq \{x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3 : x_2 \geq 1\}$, where $h(x) = x_2 - 1$ which has a relative degree 1. The initial conditions are chosen as $x(0) = \begin{bmatrix} 2 & 2.2 & 2 \end{bmatrix}^\top$, $\hat{x}(0) = \begin{bmatrix} 3 & 3.5 & 3 \end{bmatrix}^\top$, the parameters are selected as $E = 0.1$, $N = 3$, $\bar{\theta}_i = 0.5$, $\hat{\theta}_i(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$, $\epsilon = 0.1$, $\mu = 3.5$, and $M(t)$ is an exponential function with $D = 2$ and $\lambda = 0.05$. Note that A is a Hurwitz matrix, such that $u = 0$ is a stabilizing controller. In this simulation, the desired controller is selected as $u_d = 0$. It can be verified that the condition (19) in Theorem 1 is satisfied. The safe controller is obtained by solving (28). The evolution of CBF h is shown in Fig. 1 (a). Also shown is the evolution of h by utilizing the estimated state $\hat{x}$ as the true state in the traditional CBF-QP in [5].

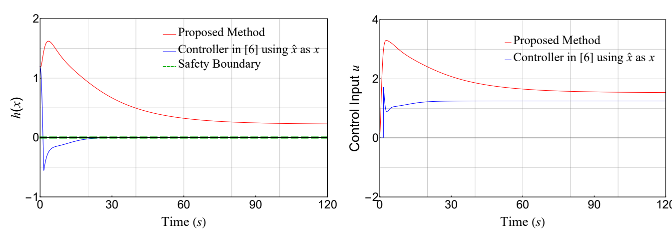
Now consider another safe set $\mathcal{C} \triangleq \{x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3 : x_1 \geq 1\}$, where the corresponding CBF is given as $h(x) = x_1 - 1$. Note that $h(x)$ has a relative degree 2. The safe controller is obtained by solving (34), where the parameters are selected as $E = 0.1$, $N = 3$, $\bar{\theta}_i = 0.5$, $\hat{\theta}_i(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$, $\epsilon = 0.1$, $\mu = 10$, and $\lambda_1 = 2$. The initial conditions are $x(0) = \begin{bmatrix} 2.4 & -3 & -3 \end{bmatrix}^\top$, $\hat{x}(0) = \begin{bmatrix} 3.4 & -2 & -2 \end{bmatrix}^\top$, and $M(t)$ is the same as above. It is easy to check the inequality (32) holds true. The simulation result is presented in Fig. 1 (b).

From the simulation results, it can be seen that regardless of the relative degree of $h(x)$, the set $\mathcal{C}$ is forward invariant when the proposed approach is used, whereas the safety constraint is violated if the estimated state, $\hat{x}$, is directly used as the true state in the traditional CBF-QP control scheme.

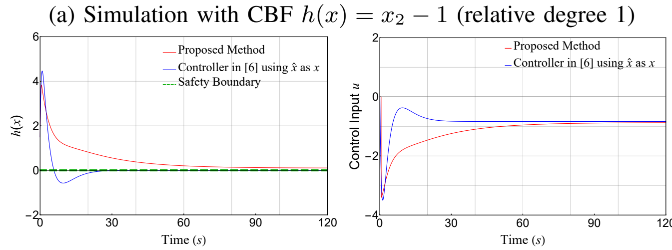
Example 2: Consider the Rössler system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_1 + a x_2 \\ b + x_3(x_1 - c) \end{bmatrix} + \boldsymbol{I}_{3\times3} u,$$

where $a = b = 0.2$, $c = 5$ are chosen as same as those in [30], $u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ denotes the control input and $\boldsymbol{I}_{3\times3}$ is the identity matrix. The exponentially stable observer in [30] is employed, where $q_1 = 3$, $s_1 = -3$, $r_1 = 3$, $q_2 = s_2 = r_2 = 10$, and $m = 3$. Consider the safe set $\mathcal{C} \triangleq \{x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3 : x_2 \geq -1\}$, where the corresponding CBF is given as $h(x) = x_2 + 1$, which has a relative degree 1. The initial conditions are given



(a) Simulation with CBF $h(x) = x_2 - 1$ (relative degree 1)



(b) Simulation with CBF $h(x) = x_1 - 1$ (relative degree 2)

Fig. 1: Simulation results of Example 1. For either CBF, the values of h is always positive when implementing the proposed CBF-QP controller, meaning that safety is always ensured; in contrast, safety is violated when utilizing the estimated state, $\hat{x}$, as the true state in the traditional CBF-QP controller.

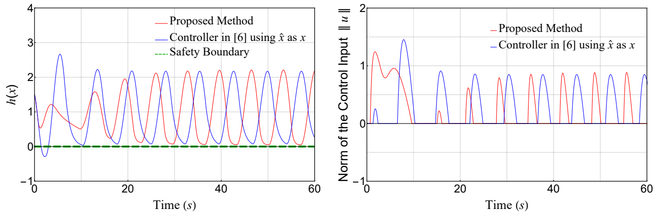


Fig. 2: Simulation results of Example 2. The CBF is $h(x) = x_2 + 1$ that has a relative degree 1. The values of h is always positive when implementing the proposed CBF-QP controller, meaning that safety is always ensured, while safety is violated when the estimated state, $\hat{x}$, is used as the true state in the traditional CBF-QP controller.

as $x(0) = \begin{bmatrix}-0.5 & 0.5 & 3\end{bmatrix}^\top$, $\hat{x}(0) = \begin{bmatrix}0.2 & 2 & 3\end{bmatrix}^\top$, the parameters are selected as $E = 0.1$, $N = 3$, $\bar{\theta}_i = 0.5$, $\hat{\theta}_i(0) = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$, $\epsilon = 0.1$, $\mu = 2.5$, and $M(t)$ is an exponential function with $D = 2$ and $\lambda = -0.15$. The desired control input is $u_d = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top$. The simulation results are depicted in Fig. 2, from which it can be seen that safety of the system is satisfied by the proposed CBF-QP controller in the presence of state estimation error while safety is violated by the traditional CBF-QP controller that uses the estimated state, $\hat{x}$, as the true state.

V. Conclusions

In this paper, a novel control framework that combines the EEQ observer and the CBF approach is proposed for safety-critical control design. The uncertainties introduced by state estimation error is approximated by FAT, and the adaptive CBF technique is employed to design controllers via quadratic programs. The proposed control strategy is validated by numerical simulations. Future studies include developing EEQ observer design techniques using deep neural networks and relaxing assumptions of this work.

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