Small-Gain Theorem for Safety Verification under High-Relative-Degree Constraints

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Abstract—**This paper develops a small-gain technique for the safety analysis and verification of interconnected systems with high-relative-degree safety constraints. To this end, a high-relative-degree input-to-state safety (ISSf) approach is proposed to quantify the influence of external inputs on the subsystem safety. With a coordination transformation, the relationship between ISSf barrier functions (ISSf-BFs) and the existing high-relative-degree (or high-order) barrier functions is established to simplify the safety analysis under external inputs. With high-relative-degree ISSf-BFs, a small-gain theorem is proposed for safety verification. It is shown that, under the small-gain condition, the compositional safe set is forward invariant and asymptotically stable. The effectiveness of the proposed small-gain theorem is illustrated on the output-constrained decentralized control of two inverted pendulums connected by a spring mounted on two carts.**

Keywords*—*Small-gain theorem, input-to-state safety, barrier functions, high relative degree, interconnected systems.

I. Introduction

Safety is a fundamental property of practical control systems, such as air traffic management systems [1], industrial robots [2], life support devices [3], and autonomous vehicles [4], [5]. Ensuring safety is crucial for these safety-critical systems. Over the past years, a set of approaches have been developed for safety verification, including model checking [8], barrier approaches [4], [6], [7], [39], and reachability analysis [1], [40].

Barrier functions have become popular because they verify safety with Lyapunov-like arguments, and avoid the computation of abstractions or reachable sets. The essential idea of barrier function approaches is to find a scalar function whose super-level set (or sub-level set, depending on the context) is forward invariant and does not intersect the unsafe region. In [6], a sum-of-square (SOS) optimization approach was developed for the computational search of a barrier function. In [4], [10], a promising barrier function, called the zeroing barrier function (ZBF), was proposed, which requires only a single super-level set of the barrier function to be invariant. Under the ZBF condition, the safe set is not only forward invariant, but also asymptotically stable, implying that the safe set is robust in the sense that the state will get into the safe set, even if it is initialized outside this set. Later, several variants of ZBFs were proposed, including nonsmooth barrier functions [11], robust barrier functions [12], and high-relative-degree (or high-order) barrier functions [13]–[16]. However, synthesizing a barrier function for high-dimensional systems remains challenging. One reason for this difficulty is that the computational cost of finding a polynomial barrier function via SOS optimization grows polynomially with respect to the system dimension for fixed polynomial degrees, as indicated in [6]. Since a complicated system can often be regarded as an interconnection of subsystems, a feasible approach is to construct barrier functions for the subsystems individually and then compose them to analyze safety for the overall interconnected system [17], [18], [20].

The small-gain technique is a fundamental tool for the analysis of interconnected systems. The classical small-gain theorem, pioneered by [21], [22], was originally established from the input-output viewpoint with linear gains. A generalization of the small-gain theorem was presented in [23] for feedback interconnections with nonlinear gains. In [24]–[26], the nonlinear small-gain theorem was developed with the help of the input-to-state stability (ISS) framework [27]. Over the past decade, the ISS small-gain theorem has been generalized to switched systems [28], hybrid systems [29], and large-scale networks [30]. Also, the small-gain theorem is useful in various control designs, such as adaptive control [31], event-triggered control [32], and output regulation [33]. Even though the small-gain theorem is important for system analysis, there are few results in safety verification.

To develop a small-gain theorem for safety verification, one must first address the characterization of subsystem safety. Different from the zero-input systems studied in the early stages of ZBFs [4], [10], the safety of an interconnected subsystem is inevitably influenced by external inputs, such as interconnection inputs from other subsystems and disturbance inputs from the environment. Under these inputs, the super-level set of a ZBF may no longer be forward invariant, and therefore, a safety buffer or margin must be added between the super-level set of the ZBF and the unsafe region. As a counterpart of ISS in safety analysis, input-to-state safety (ISSf) [34] can quantify how the safety is influence by external inputs and how much safety buffer should be added in the synthesis of a barrier function so as to handle the uncertainty of external inputs. In [17], [34], two ISSf barrier functions (ISSf-BFs) were extended from ZBFs to establish ISSf. The equivalence of these ISSf-BFs has been shown in [17]. Also, ISSf-BFs have been used in the recent paper [35] to design an inverse optimal safety-critical controller.

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Recently, small-gain theorems for safety verification have been developed in [17], [18] based on relative-degree-one ISSf-BFs (or similar concepts). Nevertheless, many practical systems have high-relative-degree safety constraints, including Euler-Largrange systems [49], automated vehicles [9], [16], [50] and pendulum systems [13], [36]. Under high-relative-degree safety constraints, it is difficult to synthesize a compositional barrier function with the techniques of [17], [18] due to the high-order derivatives involved in the subsystem barrier functions. Additionally, there are some other limitations in [17], [18]. Firstly, the subsystem barrier function used in [18] is actually a discrete-time ISS Lyapunov function defined in [48]. In this result, all super-level or sub-level sets of the compositional barrier function are required to be forward invariant. However, this requirement cannot be met in some scenarios, such as the adaptive cruise control problem in [4], [9] and the safety-critical tracking control problem in Section IV of this paper. Although [17] only requires one super-level set to be forward invariant, it has not showed whether the compositional safe set therein is asymptotically stable. Therefore, this result cannot fully inherit the robustness of ZBFs as in [4], [10].

The objective of this paper is to develop a small-gain framework for safety analysis and verification when the relative degree of safety constraints is larger than one. We aim at solving the following two fundamental problems for the small-gain framework:

- How to quantify the influence of unknown external inputs on the safe set?
- How to compose the subsystem safety to verify safety for the overall interconnected system?

The main contributions of this paper are summarized as follows.

- We propose a high-relative-degree ISSf-BF to quantify the influence of the external inputs on safe set. Additionally, we introduce a novel coordination transformation to simplify the proof of safety. Compared with the most relevant result [15], in which the barrier functions are independent of external inputs, we provide an explicit expression to describe the relationship between the invariant set and the external inputs, and prove that this set is asymptotically stable.
- We propose a small-gain theorem for the safety analysis and verification of interconnected systems with disturbance inputs and high-relative-degree safety constraints. Our result shows that, under the small-gain condition, the compositional safe set with safety buffers is forward invariant and asymptotically stable simultaneously, which implies that this invariant set inherits the robustness of ZBFs as in [4], [10]. Compared with [17], [18], our small-gain theorem is based on high-relative-degree barrier functions. Also, we allow the state to get close to the boundary of the compositional safe set, which is beneficial for safety-critical control.
- We develop a comparison lemma to prove our main result from an input-output viewpoint. In contrast to the proofs of [17], [18], we avoid the explicit construction of a compositional barrier function, which may be difficult to construct under high-relative-degree constraints. Additionally, with this lemma, we do not need to assume the interconnected system to be forward complete.

The remainder of this paper is organized as follows. In Section II, we provide an ISSf approach to quantify the influence of the unknown external inputs on the subsystem safety. Then a small-gain theorem is developed in Section III for the safety analysis and verification of feedback interconnections of ISSf subsystems. The effectiveness of this result is illustrated in Section IV with the decentralized control of two inverted pendulums connected by a spring mounted on two carts (called the pendulum-spring-cart system [36] for simplicity) with output constraints. Finally, we summarize the conclusions in Section V.

Notations. Throughout this paper, '$\circ$' denotes the composition operator, i.e., $f \circ g(s) = f(g(s))$; 'T' denotes the transpose operator; $\alpha'(s)$ denotes the derivative of a continuously differentiable function α with respect to s; $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and integers, respectively; $\mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$ denote the set of nonnegative real numbers and nonnegative integers, respectively. Given a closed set $\mathcal{S}$, denote by $\partial\mathcal{S}$ the boundary of $\mathcal{S}$. For any x in Euclidean space, $|x|$ is its norm, and $|x|_{\mathcal{S}} = \inf_{s\in\mathcal{S}} |x - s|$ denotes the point-to-set distance from x to the set $\mathcal{S}$. Denote by L^m_∞ the set of essentially bounded measurable functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. For any $u \in L^m_\infty$, $\|u\|_J$ stands for the supremum norm of u on an interval $J \subseteq \mathbb{R}_{\geq 0}$ (i.e., $\|u\|_J = \sup_{t\in J} |u(t)|$), and we take $\|u\| = \|u\|_{[0,\infty)}$ for simplicity. A continuous function γ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\gamma(0) = 0$ is of class K ($\gamma \in K$), if it is strictly increasing. A function $\gamma \in K$ is of class K_∞ ($\gamma \in K_\infty$) if $\lim_{s\to+\infty} \gamma(s) = +\infty$. A function $\beta : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class KL ($\beta \in KL$), if for each fixed t, the mapping $s \mapsto \beta(s, t)$ is of class K, and for each fixed $s \geq 0$, $t \mapsto \beta(s, t)$ is decreasing to zero as $t \to +\infty$. Since barrier functions do not have the positive definiteness of Lyapunov functions, we introduce the following extended comparison functions accordingly. A continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ with $\gamma(0) = 0$ is of extended class K ($\gamma \in EK$) if it is strictly increasing. In particular, a function $\gamma \in EK$ is of extended class K_∞ ($\gamma \in EK_\infty$) if $\lim_{s\to+\infty} \gamma(s) = +\infty$ and $\lim_{s\to-\infty} \gamma(s) = -\infty$. A function $\beta : \mathbb{R}\times\mathbb{R}_{\geq 0} \to \mathbb{R}$ is of extended class KL ($\beta \in EKL$), if for each fixed t, the mapping $s \mapsto \beta(s, t)$ is of extended class K, and for fixed $s > 0$ and $s < 0$, $t \mapsto \beta(s, t)$ is decreasing and increasing to zero, respectively, as $t \to +\infty$.

II. ISSf Under High-Relative-Degree Safety Constraints

This section presents an ISSf approach for quantifying how the safety is influenced by external inputs under high-relative-degree safety constraints.

Consider the system

$$\dot{x} = f(x, d), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $d \in L^m_\infty$ is the unknown external input (may be the interconnection input from other interconnected subsystems or the disturbance input from the environment), and $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. For any $x_0 \in \mathbb{R}^n$ and $d \in L^m_\infty$, the solution of (1), defined on some maximal interval

$I(x_0, d)$, is denoted by $x(t, x_0, d)$ (and sometimes by $x(t)$ for simplicity if there is no ambiguity). System (1) is said to have a finite escape time if $x(t, x_0, d)$ escapes to infinite at a finite time. Moreover, system (1) is forward complete if $I(x_0, d) = \mathbb{R}_{\geq 0}$.

Definition 1 (Invariance [37])**.** A set C is robustly forward invariant if, for any $x_0 \in C$ and any $d \in L_\infty^m$, $x(t, x_0, d) \in C$ for all $t \in I(x_0, d)$. Moreover, if there is no external input (i.e., $d \equiv 0$), C is said to be forward invariant.

Definition 2 (Asymptotic Stability [45])**.** System (1) is uniformly globally asymptotically stable (UGAS) with respect to a closed invariant set C if it is forward complete and the following two properties hold:

- *Uniform Stability*. There exists $\delta \in K_\infty$ such that, for any $\varepsilon \geq 0$,
$$|x(t, x_0, d)|_C \leq \varepsilon, \quad \forall t \geq 0, \quad \forall d \in L_\infty^m$$
whenever $|x_0|_C \leq \delta(\varepsilon)$.
- *Uniform Attraction*. For any $r, \varepsilon > 0$, there is a $T > 0$, such that
$$|x(t, x_0, d)|_C \leq \varepsilon, \quad \forall t \geq T, \ \forall d \in L_\infty^m$$
whenever $|x_0|_C < r$.

Moreover, if there is no external input, system (1) is said to be globally asymptotically stable (GAS) with respect to C.

Note that the uniform stability condition is equivalent to

$$|x(t, x_0, d)|_C \leq \varphi(|x_0|_C), \quad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0, \ \forall d \in L_\infty^m \qquad (2)$$

for some $\varphi \in K_\infty$. Moreover, according to [45, Proposition 2.5], system (1) is UGAS with respect to a closed and invariant set C if and only if it is forward complete and there exists $\beta \in KL$ such that

$$|x(t, x_0, d)|_C \leq \beta(|x_0|_C, t), \quad \forall x_0 \in \mathbb{R}^n, \ \forall t \geq 0, \ \forall d \in L_\infty^m. \qquad (3)$$

A. Preliminary on Barrier Functions

When there is no external input, the essential idea of safety verification based on ZBFs is to find a scalar function $h : \mathbb{R}^n \to \mathbb{R}$ for the zero-input system $\dot{x} = f(x, 0)$ such that the closed set

$$\mathcal{X} = \{x \in \mathbb{R}^n : h(x) \geq 0\} \qquad (4)$$

is forward invariant and does not intersect the unsafe region $\mathcal{X}_u$ (i.e., $\mathcal{X} \subseteq \mathbb{R}^n \backslash \mathcal{X}_u$). Such a set $\mathcal{X}$ is referred to as the safe set. As indicated in [4], [10], $\mathcal{X}$ is forward invariant and GAS if $h(x)$ is continuously differentiable and satisfies the ZBF condition[1]

$$\nabla h(x) f(x, 0) \geq -\alpha(h(x)), \quad \forall x \in \mathbb{R}^n$$

with $\alpha \in EK$. The forward invariance of $\mathcal{X}$ implies that $\dot{x} = f(x, 0)$ will be always safe if $x_0 \in \mathcal{X}$. On the other hand, by the GAS of $\mathcal{X}$,

$$|x(t, x_0, 0)|_{\mathcal{X}} \leq \beta(|x_0|_{\mathcal{X}}, t), \quad \forall x_0 \in \mathbb{R}^n, \quad \forall t \geq 0$$

for some $\beta \in KL$, which implies that, if $x_0 \notin \mathcal{X}$, then i) $x(t, x_0, 0)$ will be closer to $\mathcal{X}$ than x_0 for all $t \geq 0$; ii) $x(t, x_0, 0)$ will get into $\mathcal{X}$ as $t \to +\infty$.

Because of the external input d, the super-level set $\mathcal{X}$ of a ZBF is not forward invariant, which results in that the state $x(t, x_0, d)$ of system (1) may leave $\mathcal{X}$ and then enter the unsafe region $\mathcal{X}_u$. A promising method to address this issue is the ISSf.

Define a larger closed set

$$\mathcal{X}_d = \{x \in \mathbb{R}^n : h(x) + \varphi(\|d\|) \geq 0\} \qquad (5)$$

with $\varphi \in K_\infty$.

Definition 3 (Input-to-State Safety [34])**.** System (1) is ISSf on $\mathcal{X}$ if for any $d \in L_\infty^m$ and any x_0 in a subset of $\mathcal{X}_d$, $x(t, x_0, d)$ stays in $\mathcal{X}_d$ for all $t \in I(x_0, d)$.

As indicated in [34], if there is a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ such that

$$\nabla h(x) f(x, d) \geq -\alpha(h(x)) - \gamma(|d|), \quad \forall x \in \mathbb{R}^n \qquad (6)$$

with $\alpha \in EK_\infty$ and $\gamma \in K_\infty$, then system (1) is ISSf. The function $h(x)$ satisfying (6) is referred to as the ISSf-BF.

The ISSf implies that the state $x(t, x_0, d)$ will always stay in a larger set $\mathcal{X}_d$, although it may leave the super-level set $\mathcal{X}$ of $h(x)$. Moreover, if the external input d is bounded, the distance between $\partial\mathcal{X}$ and $\partial\mathcal{X}_d$ is also bounded. Therefore, if a safety buffer larger than $\varphi(\|d\|)$ is added in the synthesis of a barrier function $h(x)$ such that $\mathcal{X}_d$ does not intersect the unsafe region $\mathcal{X}_u$, then $x(t, x_0, d)$ will not enter $\mathcal{X}_u$.

B. ISSf Barrier Functions with High Relative Degree

Note that the ISSf-BF $h(x)$ satisfying (6) is relative-degree-one in the sense that the external input d appears if we differentiate $h(x)$ one time. In this paper, we concentrate on the situation where $h(x)$ is high-relative-degree, namely, the external input d explicitly appears until $h(x)$ is differentiated r ($r > 1$) times[2].

For any C^r function $h : \mathbb{R}^n \to \mathbb{R}$, define

$$\eta_0(x) = h(x), \quad \eta_k(x) = \dot{\eta}_{k-1}(x) + \alpha_k(\eta_{k-1}(x)), \quad 1 \leq k \leq r \qquad (7)$$

where $\alpha_k : \mathbb{R} \to \mathbb{R}$ is a C^{r-k} EK_∞ function.

Definition 4. A C^r function $h : R^n \to \mathbb{R}$ is an ISSf-BF with relative degree r for system (1), if there exist $\alpha_1, \ldots, \alpha_r \in EK_\infty$ and $\gamma \in K_\infty$ such that (7) and

$$\eta_r(x) \geq -\gamma(|d|) \qquad (8)$$

hold for all $x \in \mathbb{R}^n$ and $d \in L_\infty^m$.

The high-relative-degree ISSf-BF in Definition 4 is a variant of the ZBF [10] with the consideration of external inputs, and thus, inherits a good property of ZBFs that the state $x(t, x_0, d)$ is allowed to get close to ∂C when it is inside C, which is beneficial for safety-critical control. Moreover, a high-relative-degree ISSf-BF $h(x)$ satisfying (7) and (8) will reduce to the

[1] In this work, we assume that the barrier functions is global, namely, given a set $\mathcal{X} = \{x : h(x) \geq 0\}$, $h(x) \to +\infty$ as $|x|_{\mathbb{R}^n \backslash \mathcal{X}} \to +\infty$, and $h(x) \to -\infty$ as $|x|_{\mathcal{X}} \to +\infty$.

[2] For the simplicity of illustration, we assume that all entries of $d = [d_1, \ldots, d_m]^T$ appear after h is differentiated r times.

high-order barrier function (HOBF) of [15, Definition 2], if there is no external input and the functions $\alpha_1, \ldots, \alpha_r$ in (7) are relaxed to be EK functions. However, as indicated in the following counterexample, it may be impossible for system (1) with a non-zero input d to be ISSf, if $\alpha_1, \ldots, \alpha_r$ are relaxed to be EK functions, similar to the definition in [15].

Example 1. Consider the second-order system

$$\dot{x}_1 = -\arctan(x_1) + x_2, \quad \dot{x}_2 = -\arctan(x_2) + d \tag{9}$$

with safety constraints characterized by $\mathcal{X} = \{(x_1, x_2) : x_1 \geq 0\}$ and a bounded external input $d \equiv -\pi/2$. Let $h(x) = x_1$. Clearly, $h(x)$ satisfies (7) and (8) except that $\alpha_1(s) = \arctan(s)$ and $\alpha_2(s) = \arctan(s)$ are EK functions. Let $x_1(0) = x_2(0) = 1$. Now, we show that $x_1(t)$ and $x_2(t)$ will tend to $-\infty$. Note that $\dot{x}_2 = -\arctan(x_2) - \pi/2 < 0$, which implies that $x_2(t)$ decreases to $-\infty$ as $t \to +\infty$. By the continuity of $x_2(t)$, there is a $T \geq 0$ such that $x_2(t) \leq -\pi/2$ for all $t \geq T$. Hence, $\dot{x}_1(t) \leq -\arctan(x_1) - \pi/2 < 0$ for all $t \geq T$, and thus, $x_1(t)$ also tends to $-\infty$ as $t \to +\infty$. In summary, it is impossible for system (9) to be ISSf, even though the external input d is bounded.

Analogous to ISS Lyapunov functions (see, e.g., [38]) that have different equivalent definitions, one can redefine the high-relative-degree ISSf-BF by replacing (8) with

$$|\eta_{r-1}(x)| \geq \phi(|d|) \Rightarrow \eta_r(x) \geq 0 \tag{10}$$

for some $\phi \in K_\infty$.

Lemma 1. Inequalities (8) and (10) are equivalent.

Proof. See Appendix I. □

Let

$$\eta^*_{k-1} = -\hat{\alpha}_k \circ \gamma(\|d\|), \quad k = 1, \ldots, r$$

be the safety buffer for $\eta_{k-1}(x)$, where

$$\hat{\alpha}_k(s) = -\alpha_k^{-1} \circ \alpha_{k+1}^{-1} \circ \cdots \circ \alpha_r^{-1}(-s).$$

Take

$$\mu_k(s) = \alpha_k(s + \eta^*_{k-1}) - \alpha_k(\eta^*_{k-1}), \quad k = 1, \ldots, r. \tag{11}$$

Clearly, μ_k is of class EK_∞. Define the sets

$$\mathcal{S}_{k-1} = \{x \in \mathbb{R}^n : \eta_{k-1}(x) \geq 0\}, \tag{12}$$

$$\mathcal{C}_{k-1} = \{x \in \mathbb{R}^n : \eta_{k-1}(x) \geq \eta^*_{k-1}\}. \tag{13}$$

Recalling the set $\mathcal{X}$ in (4) and the set $\mathcal{X}_d$ in (5), we obtain $\mathcal{X} = \mathcal{S}_0$ and $\mathcal{X}_d = \mathcal{C}_0$ with $\varphi = \hat{\alpha}_1 \circ \gamma$. Moreover, $\mathcal{S} = \bigcap_{k=1}^r \mathcal{S}_{k-1}$ and $\mathcal{C} = \bigcap_{k=1}^r \mathcal{C}_{k-1}$ are the subsets of $\mathcal{X}$ and $\mathcal{X}_d$, respectively. Consider the coordination transformation

$$\tilde{\eta}_{k-1}(x) = \eta_{k-1}(x) - \eta^*_{k-1} = \eta_{k-1}(x) + \hat{\alpha}_k \circ \gamma(\|d\|) \tag{14}$$

for $k = 1, \ldots, r$. Then the relationship between the high-relative-degree ISSf-BF given in Definition 4 and the HOBF of [15, Definition 2] is established in the following lemma.

Lemma 2. Let $\mathcal{S}_{k-1}$ and $\mathcal{C}_{k-1}$ ($k = 1, \ldots, r$) be closed sets as in (12) and (13), respectively. Suppose $h(x)$ is a high-relative-degree ISSf-BF satisfying (7) and (8). Then $\tilde{h}(x) = h(x) - \eta^*_0$ is a HOBF satisfying

$$\dot{\tilde{\eta}}_{k-1}(x) = -\mu_k(\tilde{\eta}_{k-1}(x)) + \tilde{\eta}_k(x), \quad k = 1, \ldots, r-1 \tag{15}$$

$$\dot{\tilde{\eta}}_{r-1}(x) \geq -\mu_r(\tilde{\eta}_{r-1}(x)) \tag{16}$$

where μ_k and μ_r are EK_∞ functions given in (11).

The proof of this lemma is straightforward by combining (7), (8), (11), (14). The main result of this section is given as follows.

Theorem 1. Consider system (1) with safety constraints characterized by the set $\mathcal{X}$ given in (4). Let $\mathcal{S}_{k-1}$ and $\mathcal{C}_{k-1}$, $k = 1, \ldots, r$, be the sets in (12) and (13), respectively. Suppose $h : \mathbb{R}^n \to \mathbb{R}$ is an ISSf-BF with relative degree r, and satisfies (7) and (8). Then, the following properties holds:

- **(i)** system (1) is ISSf on $\mathcal{X}$, and the set $\mathcal{C} = \bigcap_{k=1}^r \mathcal{C}_{k-1}$ is robustly forward invariant;
- **(ii)** system (1) is UGAS with respect to $\mathcal{C}$ if it is forward complete.

Proof. See Appendix II. □

One essential difference between Theorem 1 and the most relevant result [15, Proposition 3] is that Theorem 1 quantifies the effect of external inputs on the invariant set $\mathcal{C}$, while [15, Proposition 3] did not consider the influence of external inputs. In other words, as can be seen in (13), our result provides an explicit expression to describe the relationship between the set $\mathcal{C}$ and the disturbance input d, and shows that this set is forward invariant and UGAS simultaneously. Another minor difference is that we assume system (1) to be forward complete, rather than assuming that $\mathcal{C}$ is compact. This is motivated by the fact that some safe sets may be non-compact (e.g., the safe set of the adaptive cruise control problem in [4], [9]), while the forward completeness of a control system can be easily guarantee by the combination of control Lyapunov functions (CLFs) and control barrier functions (CBFs) in the quadratic program-based (QP-based) safety-critical control framework [4], [9]. Also, the forward completeness assumption has been widely employed in safety verification [4], [10] and stability analysis [45], [52]. Moreover, if the safe set is compact, then the forward completeness assumption is redundant.

The proof of Theorem 1 is much more difficult than that for the relative-degree-one ISSf-BF in [34, Theorem 1] because of the high-order derivatives involved in the ISSf-BFs. Although the high-relative-degree ISSf-BF is similar to the HOBF of [15], the proof of Theorem 1 cannot be completed by the analysis of [15, Proposition 3]. This is mainly because [15, Proposition 3] requires $\eta_0(x), \ldots, \eta_{r-1}(x)$ in (7) to converge to zero, which does not hold when the external input d is non-zero. To handle this issue, we introduce the coordination transformation (14) to establish an explicit relationship between the high-relative-degree ISSf-BF in Definition 4 and the HOBF of [15]. In this way, we can prove Theorem 1 by the analysis of the auxiliary HOBF condition (15) and (16) instead of the original ISSf-BF. This analysis simplifies the proof and provides new insight for ISSf verification under high-relative-degree safety constraints.

C. ISSf Control Barrier Function with High Relative Degree

Analogous to [34], we use the following system to discuss the application of high-relative-degree ISSf-BFs in safety-critical control:

$$\dot{x} = f(x) + g(x)w \text{ with } w = u + d. \tag{17}$$

Here, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^m$ represent the control input and the disturbance entering the control channel, respectively.

Definition 5. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a C^r function and $\eta_0(x), \ldots, \eta_{r-1}(x)$ be functions satisfying (7) with $\alpha_1, \ldots, \alpha_{r-1} \in EK_\infty$. Then $h(x)$ is an ISSf control barrier function (ISSf-CBF) of relative degree r for system (17), if there exist $\alpha_r \in EK_\infty$ and $\gamma \in K_\infty$ such that

$$\sup_{u\in\mathbb{R}^m} [L_f\eta_{r-1}(x) + L_g\eta_{r-1}(x)(u + d)] \geq -\alpha_r(\eta_{r-1}) - \gamma(|d|) \tag{18}$$

holds for all $x \in \mathbb{R}^n$.

The following corollary is a direct result of Theorem 1.

Corollary 1. If system (17) has an ISSf-CBF satisfying (7) and (18), then there exists a feedback law $\psi(x)$ such that the conclusions in Theorem 1 also hold for the closed-loop system consisting of (17) and $u = \psi(x)$.

Corollary 1 implies that the ISSf-CBF controller is not sensitive to small changes in the control channel, in the sense that a bounded input d cannot result in $h(x) \to -\infty$. Thus, we can prevent the state $x(t, x_0, d)$ from entering the unsafe region $\mathcal{X}_u$ if a sufficiently large safety buffer is added in the control synthesis.

Note that the ISSf-CBF is a worst-case design method, i.e., it regards all external inputs as a factor bad for safety. However, some external inputs may be good for safety. For example, in the adaptive cruise control problem [4], [9], the wind opposing the follower car's velocity is an external input that can help the car maintain safety. In such a scenario, the ISSf-CBF will cancel the useful external input.

III. Small-Gain Theorem for Safety Verfication

The purpose of this section is to develop a small-gain theorem for the safety verification of the following interconnected system with high-relative-degree safety constraints:

$$\dot{x}_1 = f_1(x_1, x_2, d_1), \quad \dot{x}_2 = f_2(x_1, x_2, d_2), \tag{19}$$

where $x_i \in \mathbb{R}^{n_i}$ and $d_i \in L_\infty^{m_i}$ for $i = 1, 2$. Let $n = n_1 + n_2$, $x = [x_1^T, x_2^T]^T$, $x_0 = x(0)$, and $d = [d_1^T, d_2^T]^T$. Suppose that the safety constraints are characterized by the set

$$\mathcal{X} = \mathcal{X}_1 \bigcap \mathcal{X}_2 \text{ with } \mathcal{X}_i = \{x \in \mathbb{R}^n : h_i(x_i) \geq 0\} \tag{20}$$

where $h_i(x_i)$ is a C^r function.

Define

$$\eta_{i,0}(x_i) = h_i(x_i), \quad \eta_{i,k}(x_i) = \dot{\eta}_{i,k-1}(x_i) + \alpha_{i,k}(\eta_{i,k-1}(x_i)) \tag{21}$$

for $i = 1, 2$ and $k = 1, \ldots, r$, where $\alpha_{i,k}$ is a C^{r-k} EK_∞ function. Suppose that h_i is a high-relative-degree ISSf-BF for the x_i-system such that

$$\eta_{1,r}(x_1) \geq \phi_1(h_2(x_2)) - \gamma_1(|d_1|), \tag{22a}$$
$$\eta_{2,r}(x_2) \geq \phi_2(h_1(x_1)) - \gamma_2(|d_2|) \tag{22b}$$

where $\phi_i \in EK_\infty$ and $\gamma_i \in K_\infty$. Let

$$\eta^*_{i,k-1} = \min\{\hat{\phi}_{i,k}(-\hat{\gamma}_{3-i,1}(\|d\|)), -\hat{\gamma}_{i,k}(\|d\|)\} \tag{23}$$

be the safety buffer of $\eta_{i,k-1}(x_i)$, where

$$\hat{\phi}_{i,k}(s) = \alpha_{i,k}^{-1} \circ \cdots \circ \alpha_{i,r}^{-1} \circ (\mathrm{Id} + \sigma) \circ \phi_i(s) \tag{24}$$
$$\hat{\gamma}_{i,k}(s) = -\alpha_{i,k}^{-1} \circ \cdots \circ \alpha_{i,r}^{-1} \circ (\mathrm{Id} + \sigma^{-1})(-\gamma_i(s)) \tag{25}$$

for some $\sigma \in EK_\infty$. Define

$$\mathcal{S}_{i,k-1} = \{x \in \mathbb{R}^n : \eta_{i,k-1}(x_i) \geq 0\}, \tag{26}$$
$$\mathcal{C}_{i,k-1} = \{x \in \mathbb{R}^n : \eta_{i,k-1}(x_i) \geq \eta^*_{i,k-1}\} \tag{27}$$

for $i = 1, 2$ and $k = 1, \ldots, r$. Clearly, $\mathcal{X} = \mathcal{S}_{1,0} \bigcap \mathcal{S}_{2,0}$.

A. Comparison Technique

The following lemma provides a useful comparison technique for establishing the result of this section.

Lemma 3. Let $\eta : [0, T) \to \mathbb{R}$ be a continuously differentiable function such that

$$\dot{\eta}(t) \geq -\alpha(\eta(t)) + w(t), \quad \forall t \in [0, T) \tag{28}$$

with $\eta(0) = \eta_0$, where α is a locally Lipschitz EK_∞ function, and $w : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a locally essentially bounded function. Then there exists an EKL function $\beta : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $\beta(s, 0) = s$ such that

$$\eta(t) \geq \beta(\eta_0 - \eta^*, t) + \eta^*, \quad \forall t \in [0, T) \tag{29}$$

where $\eta^* = \alpha^{-1}(\inf_{t\in[0,T)} w(t))$.

Proof. See Appendix III. □

Lemma 3 differs from the standard comparison lemma (see, e.g., [46, Lemma 3.4]) in two distinct ways: i) it involves an additional function, $w(t)$, which can be used to describe the influence of external inputs on the safe set; and ii) β is an EKL function, where the first argument of β captures whether the system is initialized safely. By using Lemma 3, we can analyze the subsystem safety of an interconnected system without any forward completeness assumptions. This is useful because, as demonstrated in Example 2 below, an interconnection of two forward complete subsystems may exhibit finite escape phenomenon. Therefore, we cannot assume the interconnected system to be forward complete in advance. Additionally, the estimate in (29) is less conservative than that of [35, Definition 2], where the lower bound of η was estimated as $\eta(t) \geq \beta(\eta_0, t) + \eta^*$ for all $t \in [0, T)$.

Example 2. Consider the interconnected system

$$\dot{x}_1 = -\frac{2x_1^3}{|x_1|} + \frac{2x_2^3}{|x_2|} - 2x_1x_2, \tag{30a}$$

$$\dot{x}_2 = -\frac{x_2^3}{|x_2|} + \frac{3x_1^3}{|x_1|}. \tag{30b}$$

Suppose that system (30) needs to satisfy $x_1(t) \geq 0$ and $x_2(t) \geq 0$ simultaneously for all $t \geq 0$. Clearly, both subsystems (i.e., $\dot{x}_1 = -2x_1^3/|x_1|$ and $\dot{x}_2 = -x_2^3/|x_2|$) are safe, forward complete, and asymptotically stable with respect to the safe set when there is no interconnection. Let $x_1(0) = x_2(0) = -1$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1 + x_2) = -(x_1 + x_2)^2,$$

which implies

$$x_1(t) + x_2(t) = -\frac{2}{1-2t}.$$

Therefore, at least one of the states of (30) escapes to $-\infty$ before $t = 0.5$s.

B. Small-Gain Theorem under High-Relative-Degree Safety Constraints

This subsection presents a small-gain theorem for the safety verification of an interconnection of two ISSf subsystems with high-relative-degree constraints.

Theorem 2. Consider the interconnected system (19) with safety constraints characterized by the set $\mathcal{X}$ in (20). Let $\mathcal{S}_{i,k-1}$ and $C_{i,k-1}$ be the sets in (26) and (27), respectively. Denote by $J(x_0, d)$ the maximal interval on which the distance between $x(t)$ and $\mathbb{R}^n \backslash \mathcal{X}$ is finite. Suppose that, for $i = 1, 2$, the x_i-subsystem has a relative-degree-r ISSf-BF $h_i(x_i)$ satisfying (21) and (22). If

$$|\hat{\phi}_{1,1} \circ \hat{\phi}_{2,1}(s)| < |s|, \quad \forall s \in \mathbb{R}\backslash\{0\}, \tag{31}$$

then

- **(i)** the solution $x(t)$ is right maximally defined on $I(x_0, d) = J(x_0, d)$;
- **(ii)** system (19) is ISSf on $\mathcal{X}$, and the set $C = \bigcap_{i=1,2} \bigcap_{k=1}^{r} C_{i,k-1}$ is robustly forward invariant;
- **(iii)** system (19) is UGAS with respect to C if $J(x_0, d) = \mathbb{R}_{\geq 0}$.

Proof. See Appendix IV. □

The assumption that the distance between $x(t)$ and $\mathbb{R}^n \backslash \mathcal{X}$ is finite implies that the solution $x(t)$ is well-defined on $J(x_0, d)$ if it is inside $\mathcal{X}$. This assumption can be guaranteed by CLFs in the QP-based safety-critical control framework [4], [9].

Now, we briefly outline the main idea of proving Theorem 2 for readability. As indicated in (21) and (22), the barrier condition is a feedback loop consisting of two interconnected chains. For each chain, the output is $h_i(x_i)$, while the input is the pair $(h_{3-i}(x_{3-i}), d_i)$. Furthermore, proving forward invariance and asymptotic stability of a safe set under external inputs is essentially equivalent to analyzing how external inputs affect the boundedness and convergence of an individual ISSf-BF. This observation motivates us to prove Theorem 2 from an input-output viewpoint, which involves analyzing the boundedness and the convergence of the individual ISSf-BF $h_i(x_i)$, instead of constructing a compositional ISSf-BF. Specifically, the proof is divided into the following three steps.

- **Step 1:** For each $k = 1, \ldots, r$, apply Lemma 3 to $\dot{\eta}_{i,k-1} = -\alpha_{i,k}(\eta_{i,k-1}) + \eta_{i,k}$ to estimate how the boundedness and the convergence of $\eta_{i,k-1}$ are influenced by $\eta_{i,k}$.
- **Step 2:** For each chain, establish a relationship between its input $(h_{3-i}(x_{3-i}), d_i)$ and the boundedness or the convergence of its output $h_i(x_i)$ recursively with the boundedness or the convergence of $\eta_{i,0}, \ldots, \eta_{i,r-1}$ estimated in Step 1.
- **Step 3:** Use the small-gain condition (31) to eliminate the influence of the feedback interconnection so as to make that the boundedness and the convergence of $h_i(x_i)$ are only dependent on the disturbance input (d_1, d_2) of interconnected system (19).

For the case $\phi_1(s) \equiv 0$ or $\phi_2(s) \equiv 0$, we have $\hat{\phi}_{1,1} \circ \hat{\phi}_{2,1}(s) \equiv 0$, and thus, the small-gain condition (31) always holds. Therefore, we have the following corollary for the cascade connection of two ISSf subsystems.

Corollary 2. Consider the cascade system

$$\dot{x}_1 = f_1(x_1, x_2, d_1), \quad \dot{x}_2 = f_2(x_2, d_2) \tag{32}$$

with safety constraints characterized by the set $\mathcal{X}$ in (20). Let $J(x_0, d)$ be the maximal interval on which the distance between $x(t)$ and $\mathbb{R}^n \backslash \mathcal{X}$ is finite. Suppose that $h_1(x_1)$ and $h_2(x_2)$ are relative-degree-r ISSf-BFs satisfying (21), (22a) and

$$\eta_{2,r}(x_2) \geq -\gamma_2(|d_2|).$$

Then the conclusions of Theorem 2 also hold for system (32) with $\eta_{2,k-1}^*$ in (23) modified as $\eta_{2,k-1}^* = -\hat{\gamma}_{2,k}(\|d\|)$.

C. Relation to Other Compositional Approaches

The verification of an invariant safe set for interconnected systems has been investigated in [17]–[20]. A common feature of these results is that they are based on relative-degree-one barrier functions (or similar notions). Additionally, most of these results (except [17]) require that all super-level or sub-level sets of the compositional barrier function are forward invariant, which results in that the state cannot get close to the unsafe region, even if it is sufficiently safe.

In [19, Theorem 1], a dissipativity approach was proposed for the safety verification of interconnected systems with an arbitrary number of passive subsystems. Therein, the subsystems were assumed to have a polynomial storage function $S_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ satisfying

$$\nabla S_i(x_i) f_i(x_i, w_i, d_i) \leq w_i^T x_i - \rho_i(x_i) + \sigma_i(x_i),$$

for $i = 1, \ldots, N$, where w_i is the interconnection input used to characterize the influence of other subsystems, ρ_i is a positive-definite function, and $\sigma_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is used to capture the influence of disturbances. The idea of [19, Theorem 1] is to find a Lyapunov function $V(x) = \sum_{i=1}^{N} c_i S_i(x_i)$ satisfying $\dot{V}(x) \leq 0$ such that the safe set $C = \{x \in \mathbb{R}^n : V(x) \leq 1\}$ is forward invariant and does not intersect the unsafe region

$\mathcal{X}_u$. Here, the parameters $c_1, \ldots, c_N$ can be computed by SOS optimization. Note that [19] has not shown how the passivity is related to the subsystem safety. It is unclear how the safety of an interconnected system is influenced by the safety of its subsystems. In contrast, we use the ISSf to characterize the subsystem safety. By Theorem 2, we can establish a relationship between the safety of an interconnected system and the safety of its subsystems.

In [20, Corollary 2], another compositional approach was proposed by assuming that the subsystems have a barrier function satisfying

$$\nabla B_i(x_i) f_i(x_i, w_i, d_i) \leq \gamma_i(w_i, x_i),$$

for some functions $\gamma_i : \mathbb{R}^{m_i} \times \mathbb{R}^{n_i} \to \mathbb{R}$. It was shown that the set $C = \{x \in \mathbb{R}^n : B(x) \leq 0\}$ with $B(x) = \sum_{i=1}^N B_i(x_i)$ is forward invariant if $\sum_{i=1}^N \gamma_i(w_i, x_i) \leq 0$. An essential difference between [20, Corollary 2] and Theorem 2 is the boolean logic for composing subsystem barrier functions or subsystem safety. Specifically, [20] composed the subsystem barrier functions using the $\sum$ operator, which corresponds to the OR logical operator, as demonstrated in [51]. From this viewpoint, $B(x) = \sum_{i=1}^N B_i(x_i) \leq 0$ implies only that at least one subsystem satisfies $B_i(x_i) \leq 0$, rather than all subsystems satisfying $B_i(x_i) \leq 0$. In contrast, as can be seen in (20), we composed the subsystem safe sets $\mathcal{X}_1$ and $\mathcal{X}_1$ using the AND logic. Therefore, if the state $x(t)$ cannot leave the compositional safe set $\mathcal{X}$ defined in (20), then it also cannot leave $\mathcal{X}_1$ and $\mathcal{X}_2$.

In [18, Theorem 5.5], a discrete-time small-gain theorem was proposed for the synthesis of a compositional barrier function. Therein, the subsystem barrier function $B_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ is assumed to be non-negative and satisfy

$$B_i(f_i(x_i, w_i, u_i)) \leq \max\{\sigma_i(B_i(x_i)), \gamma_{wi}(\|w_i\|)\} \tag{33}$$

where u_i is the control input, σ_i and γ_{wi} are K_∞ functions with $\kappa_i(s) \leq s$ for all $s \geq 0$. Note that (33) is equivalent to

$$\begin{aligned} &B_i(x_i) \geq \sigma_i^{-1} \circ \gamma_{wi}(\|w_i\|) \\ &\qquad \Rightarrow B_i(f_i(x_i, w_i, u_i)) - B_i(x_i) \leq -(\mathrm{Id} - \sigma_i)(B_i(x_i)), \end{aligned}$$

which implies that $B_i(x_i)$ is an discrete-time ISS Lyapunov function originally defined in [48, Remark 3.3]. Therefore, [18, Theorem 5.5] is actually a discrete-time Lyapunov small-gain theorem. More recently, a continuous-time small-gain theorem was proposed in [17, Theorem 1] by using relative-degree-one ISSf-BFs to capture the subsystem safety. This result can be regarded as a special version of the second conclusion of Theorem 2 with $r = 1$. The essential difference between Theorem 2 and [17] or [18] lies in that our result is based on high-relative-degree ISSf-BFs, which implies we can handle the control problems with high-relative-degree safety constraints. Also, we can quantify the influence of disturbances on the compositional safe set. As shown in (23) and (27), we have derived an explicit expression to describe the relationship between the compositional invariant set $C = \bigcap_{i=1,2} \bigcap_{k=1}^r C_{i,k-1}$ and the disturbance input d, and showed that C is UGAS, implying that Theorem 2 inherits the robustness of ZBFs as in [4], [10]. This is different from [18, Theorem 5.5], where no disturbance was involved, and [17, Theorem 1], where only forward invariance was proved. Finally, the idea of proving Theorem 2 is also different from [17, Theorem 1] and [18, Theorem 5.5] whose proofs are based on the construction of a compositional barrier function. In contrast, our proof relies on the boundedness and convergence analysis of subsystem barrier functions, and thus, avoids an explicit construction of a compositional barrier function.

IV. Illustrative Example

In this section, we illustrate the effectiveness of the proposed small-gain technique through the decentralized tracking control of the pendulum-spring-cart system [36, Section 7]:

$$\dot{x}_{i,1} = x_{i,2} \tag{34a}$$

$$\begin{aligned} \dot{x}_{i,2} = &\frac{g}{wl} x_{i,1} - \frac{m}{M} x_{i,2}^2 \sin x_{i,1} - \frac{ak(a - wl)}{wml^2} x_{i,1} \\ &+ \frac{kb(a - wl)}{wml^2} + \frac{1}{wml^2}(u_i + d_i) + \frac{ak(a - wl)}{wml^2} x_{3-i,1} \end{aligned} \tag{34b}$$

for $i = 1, 2$, where $x_{i,1} = \theta_i$ and $x_{i,2} = \dot{\theta}_i$ denote the angular displacement and the angular velocity, respectively, u_i is the control torque applied to the pendulum, d_i is the disturbance in the control channel with $|d_i| \leq \bar{d}$, m and l are the mass and the length of the pendulum, M is the mass of the car, $w = m/(M + m)$, k is the spring constant, L is natural length of the spring, $a \in [0, l]$ is the distance from the pivot of the spring to the bottom of the pendulum, g is the gravitational acceleration, and b is the distance between the cars. Choose $g = 9.8$ m/s^2, $l = 1$ m, $k = 1$ n/m. $M = 15$ kg, $m = 5$ kg, $b = 2$ m and $a = 0.75$ m.

Suppose that the safety constraint of pendulum i is $\theta_i(t) \geq \underline{\theta}_i$, where $\underline{\theta}_i$ denotes the lower bound of $\theta_i(t)$. The goal is to make the output $\theta_i(t)$ of the pendulum track its own reference trajectory $y_{r,i}$, while simultaneously avoiding the violation of safety constraints.

A. Nominal Tracking Controller

We design a nominal tracking controller with the backstepping technique [41]. Consider the coordination transformation

$$z_{i,1} = x_{i,1} - y_{r,i}, \quad z_{i,2} = x_{i,2} - \varpi_i$$

where $\varpi_i = -r_{i,1} z_{i,1} + \dot{y}_{r_i}$ with $r_{i,1} > 0$ as a designed parameter. Then the nominal controller is chosen as

$$\begin{aligned} \hat{u}_i = wml^2 \Big(&- r_{i,2} z_{i,2} - W_i - \frac{1}{4wml^2} z_{i,2}^2 \\ &- \frac{ak(a - wl)}{wml^2}(z_{i,2} + y_{r,3-i}) \Big) \end{aligned} \tag{35}$$

where $r_{i,2} > 0$ is a designed parameter, and

$$\begin{aligned} W_i = &z_{i,1} - \dot{\varpi}_i + \frac{g}{wl} x_{i,1} - \frac{m}{M} x_{i,2}^2 \sin x_{i,1} \\ &- \frac{ak(a - wl)}{wml^2} x_{i,1} + \frac{kb(a - wl)}{wml^2}. \end{aligned}$$

Clearly, the derivative of the Lyapunov function candidate $V_i = (z_{i,1}^2 + z_{i,2}^2)/2$ along the solution of the closed-loop system consisting of (34) and (35) satisfies

$$\dot{V}_i \leq -\lambda_i(V_i) + \varphi_i(V_{3-i}) + \delta_i(|d|)$$

where $\lambda_i(s) = \min\{r_{i,1}, r_{i,2}\}s$, $\varphi_i(s) = \frac{ak(a-wl)}{2wml^2}s$, $\delta_i(s) = \frac{1}{wml^2}s^2$. Choose sufficiently large λ_i for $i = 1, 2$, such that $\lambda_1^{-1} \circ \varphi_1 \circ \lambda_2^{-1} \circ \varphi_2(s) < s$ for all $s > 0$, and according to [42, Theorem 5.1], the tracking error $z_{i,1}$ is driven to a small neighbourhood of zero.

B. Control Barrier Function

Let $h_i(x_i) = x_{i,1} - \underline{\theta}_i$, which is clearly with relative degree two. Then we can establish (21) and (22) with

$$\eta_{i,0}(x_i) = h_i(x_i)$$
$$\eta_{i,1(x_i)} = \dot{\eta}_{i,0}(x_i) + \alpha_{i,1}(h_i)$$
$$\eta_{i,2}(x_i) \geq \psi_{i,1}(x_i) + \psi_{i,0}(x_i)u_i + \phi_i(h_{3-i}(x_{3-i})) - \gamma_i(|d_i|).$$

where

$$\psi_{i,1}(x_i) = \frac{g}{wl}x_{i,1} - \frac{m}{M}x_{i,2}^2 \sin x_{i,1} + \alpha'_{i,1}(h_i)x_{i,2} + \alpha_{i,2}(\eta_{i,1}) - \frac{k(a-wl)}{wml^2}(ax_{i,1} - b - \underline{\theta}_{3-i}),$$
$$\psi_{i,0}(x_i) = \frac{1}{wml^2}, \quad \phi_i(s) = \frac{ak(a-wl)}{wml^2}s, \quad \gamma_i(s) = \frac{1}{wml^2}s.$$

Herein, $\alpha_{i,1}$ and $\alpha_{i,2}$ are EK_∞ functions that can be tuned by designers. Inspired by [4], [9], [10], [34], [43], any control input u_i in the set

$$U_i = \{u_i \in \mathbb{R} : \psi_{i,1}(x_i) + \psi_{i,0}(x_i)u_i \geq 0\} \tag{36}$$

renders

$$\eta_{i,2}(x_i) \geq \phi_i(h_{3-i}(x_{3-i})) - \gamma_i(|d_i|).$$

Take $\alpha_{i,k}(s) = c_{i,k}s$ for $i = 1, 2$ and $k = 1, 2$. Select a sufficiently large $c_{i,k}$ such that (31) is satisfied. Then, according to Theorem 2, the set $C = \bigcap_{i=1,2}\bigcap_{k=1,2} C_{i,k-1}$ with $C_{i,k-1}$ defined in (27) is forward invariant. Thus, $\eta_{i,k-1}(x_i(t)) \geq \eta_{i,k-1}^*$ always holds for all $t \geq 0$ if $x(0) \in C$. Moreover, if there is no disturbance, the system always satisfies the safety constraint $h_1(x_1) \geq 0$ and $h_2(x_2) \geq 0$. On the other hand, if $x(0) \notin C$, $x(t)$ will get into C as $t \to +\infty$.

C. Simulation Results

With (35) and (36), we can establish the QP-based controller as in [4], [9], [34]:

$$u_i^* = \arg\min_{u_i \in \mathbb{R}} |u_i - \hat{u}_i|^2,$$
$$\text{s.t. } \psi_{i,1}(x_i) + \psi_{i,0}(x_i)u_i \geq 0.$$

We first study the system performance when there is no disturbance. Set $\underline{\theta}_1 = \underline{\theta}_2 = -0.3$, $y_{r,1} = -\sin(t + \pi/4)$ and $y_{r,2} = \sin(t)$. Fig. 1 illustrates the trajectories of $x_{1,1}(t)$ and $x_{2,1}(t)$ with a safe initialization condition $(x_{1,1}(0), x_{1,2}(0)) = (x_{2,1}(0), x_{2,2}(0)) = (0.5, 1.0)$ and control parameters $r_{1,1} = r_{2,1} = 4$, $r_{1,2} = r_{2,2} = 2$, $c_{1,1} = c_{2,1} = 12$, and $c_{1,2} = c_{2,2} = 4$. Clearly, $x_{1,1}(t)$ and $x_{2,1}(t)$ always stay inside the safe region, and moreover, the tracking task is achieved if the reference signal is inside the safe region. Fig. 2 shows the simulations of $x_{1,1}(t)$ under the same control parameters of Fig. 1 but $(x_{1,1}(0), x_{1,2}(0)) = (-0.8, 1.0)$. As indicated in the time interval from $t = 0$ s to $t = 1$ s, even though $x_{1,1}(t)$ is initialized near the reference signal, it tends to the safe region at first, rather than tracking the reference signal. Then it does not violate the safety constraint any more after entering the safe region. In contrast, the safety cannot be guaranteed after removing the CBF constraints (see the green dash line in Fig. 2).

Now we check how the disturbance input affects the system performance. Let $d = (d_1, d_2)$. The simulations of $x_{1,1}(t)$ under different disturbance inputs are given in Fig. 3. It is obvious that the existence of disturbances degrades the tracking performance. Additionally, using (23), we can compute that $\eta_{1,1}^* \approx -0.169$ and $\eta_{1,1}^* \approx -0.337$ for $d \equiv (-10, -10)$ and $d \equiv (-20, -20)$, respectively. As can be seen in Fig. 3, when the disturbance is non-zero, $x_{1,1}$ violates the safety constraint $x_{1,1} \geq \underline{\theta}_1$, but always satisfies $x_{1,1} \geq \underline{\theta}_1 + \eta_1^*$. Therefore, if one has the opportunity to redesign the controller, it is possible to ensure safety by adding sufficient large safety buffer in the synthesis of a barrier function. For example, if the disturbance bound is known, one can take $\tilde{h}_i(x_i) = x_{i,1} - \underline{\theta}_i + \eta_{1,1}^*$ as a new barrier function.

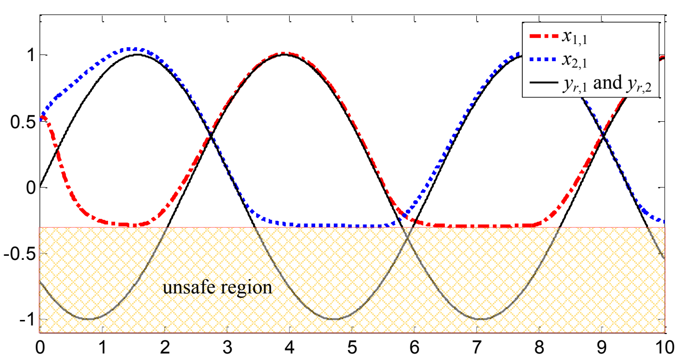

Fig. 1. Tracking results of $x_{1,1}(t)$ and $x_{2,1}(t)$ under $d \equiv (0, 0)$.

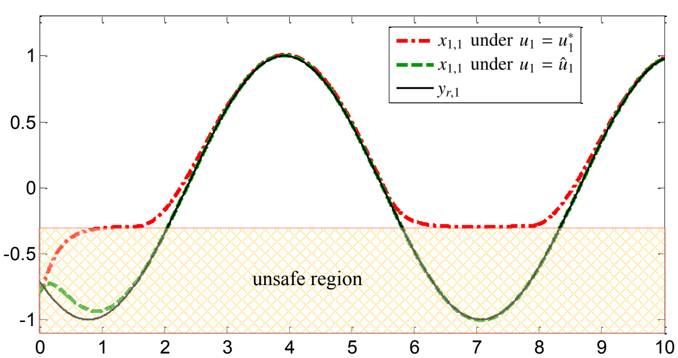

Fig. 2. Tracking results of $x_{1,1}(t)$ under $d \equiv (0, 0)$ with different control laws.

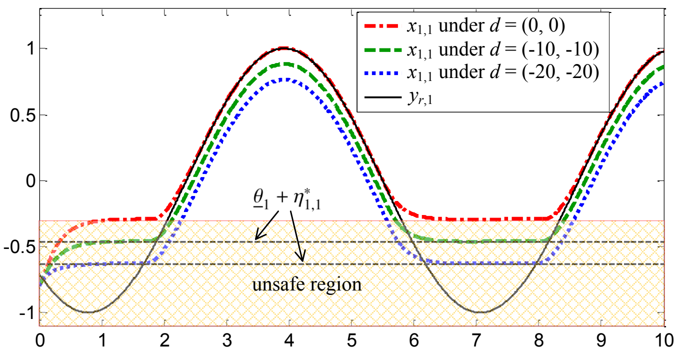

Fig. 3. Tracking results of $x_{1,1}(t)$ under different disturbance inputs.

V. Conclusions

In this work, we developed a small-gain technique for the safety verification of interconnected systems under high-relative-degree safety constraints. We proposed a high-relative-degree ISSf approach to quantify the influence of external inputs on the subsystem safety. A relationship between the high-relative-degree ISSf-BFs and HOBFs was given to simplify the ISSf analysis. With the help of high-relative-degree ISSf-BFs, a small-gain theorem was developed for the safety analysis and verification of interconnected systems. Finally, the decentralized control of a pendulum-spring-cart system with output constraints was used to illustrate the effectiveness of our result.

Appendix I: Proof of Lemma 1

(8) $\Rightarrow$ (10). According to (7) and (8),

$$\eta_{r-1}(x) \le \alpha_r^{-1}(-\gamma(|d|)/c) \Rightarrow \dot{\eta}_{r-1}(x) + (1-c)\alpha_r(\eta_{r-1}(x)) \ge 0,$$
$$\eta_{r-1}(x) \ge \alpha_r^{-1}(\gamma(|d|)/c) \Rightarrow \dot{\eta}_{r-1}(x) + (1+c)\alpha_r(\eta_{r-1}(x)) \ge 0$$

where $c \in (0,1)$ is a constant. Let

$$\phi(s) = \max\{-\alpha_r^{-1}(-\gamma(s)/c), \alpha_r^{-1}(\gamma(s)/c)\},$$
$$\sigma(s) = \max\{(1-c)\alpha_r(s), (1+c)\alpha_r(s)\}.$$

Clearly, ϕ is a K_∞ function, and σ is an EK_∞ function. Thus, (10) follows by taking $\eta_r(x) = \dot{\eta}_{r-1}(x) + \sigma(\eta_{r-1}(x))$.

(10) $\Rightarrow$ (8). According to (10), if $|\eta_{r-1}(x)| \le \phi(|d|)$, then

$$\begin{aligned}\eta_r(x) &= \alpha_r(\eta_{r-1}(x)) + \nabla\eta_{r-1}(x)f(x,d)\\ &\ge \alpha_r(-\phi(|d|)) + \nabla\eta_{r-1}(x)f(x,d)\\ &\ge -\gamma(|d|)\end{aligned}$$

where

$$\gamma(s) = -\alpha_r(-\phi(s)) - \inf_{|\eta_{r-1}(x)|\le\phi(s),|d|\le s} \min\{0, \nabla\eta_{r-1}(x)f(x,d)\};$$

on the other hand, if $|\eta_{r-1}(x)| \ge \phi(|d|)$, then $\eta_r(x) \ge 0 \ge -\gamma(|d|)$. The rest is to show $\gamma \in K_\infty$. Because ϕ is a K_∞ function, the set $\{x \in \mathbb{R}^n : |\eta_{r-1}(x)| \le \phi(s)\}$ is compact for fixed $s \ge 0$, and thus, the term $\inf_{|\eta_{r-1}(x)|\le\phi(s),|d|\le s} \min\{0, \nabla\eta_{r-1}(x)f(x,d)\}$ is well-defined, non-negative and non-increasing for all $s \ge 0$. Moreover, because $\alpha_r \in EK_\infty$ and $\phi \in K_\infty$, $\gamma(s)$ is strictly increasing and tends to $+\infty$ as $s \to +\infty$. Additionally, with (10), we have $\eta_r(x) = \nabla\eta_{r-1}(x)f(x,0) + \alpha_r(\eta_{r-1}(x)) \ge 0$ whenever $|d| = 0$, which implies that $\inf_{|\eta_{r-1}(x)|\le\phi(s),|d|\le s} \min\{0, \nabla\eta_{r-1}(x)f(x,d)\}$ is zero at $s = 0$. Consequently, γ is a K_∞ function.

Appendix II: Proof of Theorem 1

A. Proof of (i) of Theorem 1

By applying [15, Proposition 1] to the auxiliary HOBF condition (15) and (16), we have

$$\tilde{\eta}_{k-1}(x(t)) = \eta_{k-1}(x(t)) - \eta^*_{k-1} \ge 0, \ \forall t \ge 0, \ \forall x_0 \in C$$

which implies that $x(t)$ always stay in the set C_{k-1} for each for $k = 1, \dots, r$. Therefore, C is robustly forward invariant. Because C is a subset of $\mathcal{X}_d = C_0$, $x(t, x_0, d)$ cannot leave $\mathcal{X}_d$ for all $x_0 \in C$, which further implies the ISSf of system (1) on the set $\mathcal{X} = \mathcal{S}_0$.

B. Proof of (ii) of Theorem 1

Let

$$\tilde{V}_{k-1}(x) = \max\{0, -\tilde{\eta}_{k-1}(x)\}, \quad k = 1\dots, r. \tag{37}$$

Since $-\tilde{\eta}_{k-1}(x) \le 0$ whenever $x \in C_{k-1}$, (37) is equivalent to

$$\tilde{V}_{k-1}(x) = \begin{cases} 0, & \text{if } x \in C_{k-1};\\ -\tilde{\eta}_{i,k-1}(x_i), & \text{if } x \in \mathbb{R}^n\backslash C_{k-1}.\end{cases}$$

Because $\tilde{V}_{k-1} = \max\{0, -\tilde{\eta}_{k-1}\} \ge -\tilde{\eta}_{k-1}$, it follows from (15) and (16) that

$$\dot{\tilde{V}}_{k-1}(x) \le \mu_k(-\tilde{V}_{k-1}(x)) + \tilde{V}_k(x), \quad k = 1, \dots, r-1, \tag{38}$$
$$\dot{\tilde{V}}_{r-1}(x) \le \mu_r(-\tilde{V}_{r-1}(x)). \tag{39}$$

Consider the comparison system

$$\underbrace{\begin{bmatrix} \dot{m}_0\\ \dot{m}_1\\ \cdots\\ \dot{m}_{r-1}\end{bmatrix}}_{\dot{m}} = \underbrace{\begin{bmatrix} \mu_1(-m_0) + m_1\\ \mu_2(-m_1) + m_2\\ \cdots\\ \mu_r(-m_{r-1})\end{bmatrix}}_{F(m)} \tag{40}$$

with $[m_0(0), \dots, m_{r-1}(0)]^T = [\tilde{V}_0(x(0)), \dots, \tilde{V}_{r-1}(x(0))]^T$. Because the vector field F is quasi-monotone increasing[3], by the vectorial comparison lemma (see, e.g., Lemma 2.3 of [44, Chapter 9]), $\tilde{V}_{k-1}(x(t)) \le m_{k-1}(t)$ for all $t \ge 0$ with $k = 1, \dots, r$. Moreover, from [15, Proposition 3], system (40) is asymptotically stable. Let $\tilde{V}(x) = \max_{k=1,\dots,r} \tilde{V}_{k-1}(x)$. With [45, Proposition 2.5], there exists $\beta \in KL$ such that

$$\tilde{V}(x(t)) \le |m(t)| \le \beta(\tilde{V}(x_0), t), \quad \forall t \ge 0. \tag{41}$$

Take

$$\underline{\psi}(s) = \inf_{|x|_C\ge s} \tilde{V}(x), \quad \bar{\psi}(s) = \sup_{|x|_C\le s} \tilde{V}(x), \quad \forall s \ge 0.$$

[3]As indicated in [44, p.314], a vector field $F : \mathbb{R}^r \to \mathbb{R}^r$ is said to be quasi-monotone increasing, if $F_k(x) \ge F_k(y)$ for every $k = 1, \dots, r$ and any two points $x, y \in \mathbb{R}^r$ satisfying i) $x_p = y_p$ if $p = k$, and ii) $x_p \ge y_p$ if $p \ne k$. Herein, the subscript represents the index of entries.

Note that $V(x)$ is zero inside C, positive for all $x \in \mathbb{R}^n \backslash C$, and tends to $+\infty$ as $|x|_C \to +\infty$. Thus, $\underline{\psi}$ and $\bar{\psi}$ are continuous, nondecreasing and unbounded on $\mathbb{R}_{\geq 0}$, and satisfy $\underline{\psi}(0) = \bar{\psi}(0) = 0$. Choose functions $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(s) \leq \underline{\psi}(s)/c$ and $\bar{\alpha}(s) \geq c\bar{\psi}(s)$ with $c > 1$. Therefore,

$$\underline{\alpha}(|x|_C) \leq \underline{\psi}(|x|_C) \leq V(x) \leq \bar{\psi}(|x|_C) \leq \bar{\alpha}(|x|_C). \tag{42}$$

Then, with (41),

$$|x(t)|_C \leq \underline{\alpha}^{-1}(\beta(\bar{\alpha}(|x_0|_C), t)), \quad \forall t \geq 0, \tag{43}$$

which, together with (3), implies that system (1) is UGAS with respect to C.

Appendix III: Proof of Lemma 3

From (28),

$$\dot{\eta}(t) \geq -\alpha(\eta(t)) + \alpha(\eta^*), \quad \forall t \in [0, T). \tag{44}$$

Consider its comparison equation

$$\dot{y} = -\alpha(y) + \alpha(\eta^*), \quad y(0) = \eta_0. \tag{45}$$

Claim 1. The comparison equation (45) has a unique solution $y(t)$ defined on $\mathbb{R}_{\geq 0}$. Moreover,

$$y(t) = \beta(y_0 - \eta^*, t) + \eta^* \tag{46}$$

where $\beta : \mathbb{R}\times\mathbb{R}_{\geq 0} \to \mathbb{R}$ is of class $\mathcal{EKL}$ and satisfies $\beta(s, 0) = s$.

Then the conclusion of Lemma 3 follows, by applying Claim 1 and the standard comparison lemma [46, Lemma 3.4] to (44). Thus, the rest is to prove this claim.

Proof of Claim 1. The local Lipschitzness of α implies that (45) has a unique solution $y(t)$ for each $y_0 \in \mathbb{R}$. Since $y = \eta^*$ is an equilibrium point of (45) and $\dot{y}(t) < 0$ (resp. $\dot{y}(t) > 0$) when $y(t) > \alpha^{-1}(w)$ (resp. $y(t) < \alpha^{-1}(w)$), it follows that $-|y_0| \leq y(t) \leq |y_0|$. Therefore, the solution of (45) is bounded and can be extended indefinitely.

Take $\tilde{y} = y - \eta^*$, and then (45) can be rewritten as

$$\dot{\tilde{y}} = -\hat{\alpha}(\tilde{y}), \quad \tilde{y}(0) = y_0 - \eta^* \tag{47}$$

where $\hat{\alpha}(s) = \alpha(s + \eta^*) - \alpha(\eta^*)$ with $\hat{\alpha}(0) = 0$ is also a locally Lipschitz $\mathcal{EK}_\infty$ function. Note that $\tilde{y}(t) \equiv 0$ if $\tilde{y}_0 = 0$, since $\tilde{y} = 0$ is an equilibrium of (47). Without loss of generality, we assume $\tilde{y}_0 \neq 0$ in the following. By integration, the solution $\tilde{y}(t)$ of (47) satisfies

$$-\int_{\tilde{y}(0)}^{\tilde{y}(t)} \frac{\mathrm{d}r}{\hat{\alpha}(r)} = \int_0^t \mathrm{d}\tau. \tag{48}$$

Define, for any $s \in \mathbb{R}\backslash\{0\}$,

$$\eta(s) = \begin{cases} -\int_1^s \frac{dr}{\hat{\alpha}(r)}, & \text{if } s > 0 \\ -\int_{-1}^s \frac{dr}{\hat{\alpha}(r)}, & \text{if } s < 0 \end{cases} \tag{49}$$

which is strictly decreasing on $(0, +\infty)$ and strictly increasing on $(-\infty, 0)$. From the uniqueness of the solution of (47), it follows that $\tilde{y}(t)$ tends to zero if and only if t tends to infinity, and thus, $\tilde{y}(t) \geq 0$ (resp. $\tilde{y}(t) \leq 0$) for all $t \geq 0$ if $\tilde{y}_0 \geq 0$ (resp. $\tilde{y}_0 \leq 0$). Recalling (48) and (49), the solution $\tilde{y}(t)$ of (47) satisfies

$$\eta(\tilde{y}(t)) - \eta(\tilde{y}(0)) = t.$$

Let

$$\beta(s, t) = \begin{cases} \eta^{-1}(\eta(s) + t), & \text{if } s \neq 0, \\ 0, & \text{if } s = 0. \end{cases}$$

Then $\tilde{y}(t) = \beta(\tilde{y}(0), t)$, which implies (46) for all $t \geq 0$. The rest is to show that β is of class $\mathcal{EKL}$. Since $\hat{\alpha}$ is locally Lipschitz, for each $s \in \mathbb{R}\backslash 0$, $|\hat{\alpha}(s)| \leq K|s|$. Consequently,

$$\lim_{s\to 0^+} \eta(s) = \lim_{s\to 0^+} \int_s^1 \frac{dr}{\hat{\alpha}(r)} \geq \lim_{s\to 0^+} \int_s^1 \frac{dr}{Kr} = +\infty,$$
$$\lim_{s\to 0^-} \eta(s) = -\lim_{s\to 0^-} \int_{-1}^s \frac{dr}{\hat{\alpha}(r)} \geq -\lim_{s\to 0^-} \int_{-1}^s \frac{dr}{Kr} = +\infty.$$

As a result,

$$\lim_{s\to +\infty} \eta^{-1}(s) = 0.$$

Since η and η^{-1} are continuous functions, β is also continuous. For each fixed $t \geq 0$,

$$\frac{\partial}{\partial s}\beta(s, t) = \frac{\eta'(s)}{\eta'(\beta(s, t))} = \frac{\hat{\alpha} \circ \eta^{-1}(\eta(s) + t)}{\hat{\alpha}(s)} > 0,$$

and thus, β is strictly increasing on s. In addition,

$$\frac{\partial}{\partial t}\beta(s, t) = \frac{1}{\eta'(\beta(s, t))} = -\hat{\alpha} \circ \eta^{-1}(\eta(s) + t).$$

Therefore, $\partial\beta(s, t)/\partial t < 0$ for each fixed $s > 0$ and $\partial\beta(s, t)/\partial t > 0$ for each $s < 0$. Because $\lim_{s\to +\infty} \eta^{-1}(s) = 0$, $\beta(s, t)$ decreases and increases to zero for each fixed $s > 0$ and $s < 0$, respectively, as t tends to $+\infty$. □

Appendix IV: Proof of Theorem 2

In order to prove Theorem 2, we introduce a useful inequality, that is, for any functions $\gamma, \sigma \in \mathcal{EK}_\infty$ and any real numbers a, b,

$$\gamma(a + b) \geq \min\{\gamma \circ (\mathrm{Id} + \sigma)(a), \gamma \circ (\mathrm{Id} + \sigma^{-1})(b)\}. \tag{50}$$

This inequality is extended from [24, Inequality (6)] by removing the positive definiteness assumption. It can be verified by combining the following two cases: i) if $b \geq \sigma(a)$, then $\gamma(a + b) \geq \gamma \circ (\mathrm{Id} + \sigma)(a)$; and ii) if $b \leq \sigma(a)$, then $\gamma(a + b) \geq \gamma \circ (\mathrm{Id} + \sigma^{-1})(b)$.

A. Proof of (i) of Theorem 2

Suppose that, for any $T \in J(x_0, d)$, the solution $x(t)$ of system (19) is right maximally defined on $[0, T)$. Let

$$V_{i,k-1}(x_i) = \max\{0, -\eta_{i,k-1}(x_i)\} \tag{51}$$

for $i = 1, 2$ and $k = 1, \ldots, r$. From (21) and (26), we have the following two cases:

- if $x_i \in \mathcal{S}_{i,k-1}$, $V_{i,k-1}(x_i) = 0$, and thus,

$$\dot{V}_{i,k-1}(x_i) = 0 \leq \alpha_{i,k}(-V_{i,k-1}(x_i)) + V_{i,k}(x_i);$$

- if $x_i \notin \mathcal{S}_{i,k-1}$, $V_{i,k-1}(x_i) = -\eta_{i,k-1}(x_i)$, and thus, by combining (21) and (51),

$$\begin{aligned}\dot{V}_{i,k-1}(x_i) &= -\dot{\eta}_{i,k-1}(x_i)\\ &= \alpha_{i,k}(\eta_{i,k-1}(x_i)) - \eta_{i,k}\\ &\leq \alpha_{i,k}(\eta_{i,k-1}(x_i)) + \max\{0, -\eta_{i,k}\}\\ &= \alpha_{i,k}(-V_{i,k-1}(x_i)) + V_{i,k}.\end{aligned}$$

Consequently,

$$\dot{V}_{i,k-1}(x_i) \leq \alpha_{i,k}(-V_{i,k-1}(x_i)) + V_{i,k}(x_i), \quad \forall x_i \in \mathbb{R}^{n_i}. \tag{52}$$

By applying Lemma 3 to (52), with taking $\eta = -V_{i,k-1}$, $\alpha = \alpha_{i,k}$, and $w = -V_{i,k}$, there is an EKL function $\beta_{i,k}$ satisfying $\beta_{i,k}(s, 0) = s$ such that

$$\begin{aligned}&V_{i,k-1}(x_i(t))\\ &\quad\leq \beta_{i,k}\Big(V_{i,k-1}(0) + \alpha_{i,k}^{-1}(-\|V_{i,k}\|_{[0,T)}), t\Big) - \alpha_{i,k}^{-1}(-\|V_{i,k}\|_{[0,T)}).\end{aligned} \tag{53}$$

Since the mapping $t \mapsto \beta_{i,k}(s, t)$ is strictly increasing (or decreasing) for each $s < 0$ (or $s > 0$),

$$\|V_{i,k-1}\|_{[0,T)} \leq \max\{V_{i,k-1}(0), -\alpha_{i,k}^{-1}(-\|V_{i,k}\|_{[0,T)})\} \tag{54}$$

for $k = 1, \ldots, r$. On the other hand, by combining (22) and (50), there is a $\sigma \in EK_\infty$ such that

$$\eta_{i,r}(x_i) \geq \min\{(\mathrm{Id} + \sigma) \circ \phi_i(\eta_{3-i,0}(x_{3-i})), (\mathrm{Id} + \sigma^{-1})(-\gamma_i(|d_i|))\}.$$

Therefore,

$$\begin{aligned}\|V_{i,r}\|_{[0,T)} &= \sup_{t\in[0,T)} \max\{0, -\eta_{i,r}(x_i(t))\}\\ &\leq \max\{-(\mathrm{Id} + \sigma) \circ \phi_i(-\sup_{t\in[0,T)} -\eta_{3-i,0}(x_{3-i}(t))),\\ &\qquad - (\mathrm{Id} + \sigma^{-1})(-\gamma_i(\|d\|))\}\\ &\leq \max\{-(\mathrm{Id} + \sigma) \circ \phi_i(-\|V_{3-i,0}\|_{[0,T)}),\\ &\qquad - (\mathrm{Id} + \sigma^{-1})(-\gamma_i(\|d\|))\}.\end{aligned} \tag{55}$$

Combining (54) and (55),

$$\begin{aligned}\|V_{i,0}\|_{[0,T)} &\leq \max\{V_{i,0}(0), -\alpha_{i,1}^{-1}(-\|V_{i,1}\|_{[0,T)})\}\\ &\leq \max\{V_{i,0}(0), -\alpha_{i,1}^{-1}(-V_{i,1}(0)), -\alpha_{i,1}^{-1} \circ \alpha_{i,2}^{-1}(-\|V_{i,2}\|_{[0,T)})\}\\ &\cdots\\ &\leq \max\{\Delta_{i,0}, -\hat{\phi}_{i,1}(-\|V_{3-i,0}\|_{[0,T)}), \hat{\gamma}_{i,1}(\|d\|)\},\end{aligned}$$

where

$$\begin{aligned}\Delta_{i,0} = \max\{&V_{i,0}(0), -\alpha_{i,1}^{-1}(-V_{i,1}(0)), \ldots,\\ &- \alpha_{i,1}^{-1} \circ \cdots \circ \alpha_{i,r-1}^{-1}(-V_{i,r-1}(0))\}.\end{aligned}$$

Hence,

$$\begin{aligned}\|V_{1,0}\|_{[0,T)} &\leq \max\{\Delta_{1,0}, -\hat{\phi}_{1,1}(-\Delta_{2,0}), -\hat{\phi}_{1,1} \circ \hat{\phi}_{2,1}(-\|V_{1,0}\|),\\ &\qquad - \hat{\phi}_{1,1}(-\hat{\gamma}_{2,1}(\|d\|)), \hat{\gamma}_{1,1}(\|d\|)\}\\ &= \max\{\Delta_{1,0}, -\hat{\phi}_{1,1}(-\Delta_{2,0}),\\ &\qquad - \hat{\phi}_{1,1} \circ \hat{\phi}_{2,1}(-\|V_{1,0}\|), -\eta_{1,0}^*\}\end{aligned}$$

with $\eta_{1,0}^*$ given in (23). With the small-gain condition (31),

$$\|V_{1,0}\|_{[0,T)} \leq \max\{\Delta_{1,0}, -\hat{\phi}_{1,1}(-\Delta_{2,0}), -\eta_{1,0}^*\}. \tag{56}$$

By the symmetry between $V_{1,0}$ and $V_{2,0}$,

$$\|V_{2,0}\|_{[0,T)} \leq \max\{\Delta_{2,0}, -\hat{\phi}_{2,1}(-\Delta_{1,0}), -\eta_{2,0}^*\}. \tag{57}$$

Because T is arbitrary on $J(x_0, d)$ and the right-hand sides of (56) and (57) are independent of T, we have

$$\|V_{i,0}\|_{J(x_0,d)} \leq \max\{\Delta_{i,0}, -\hat{\phi}_{i,1}(-\Delta_{3-i,0}), -\eta_{i,0}^*\}, \quad i = 1, 2.$$

Analogous to the derivation of (42), since $V_{i,0}$ is zero in the set $\mathcal{S}_{i,0}$, positive in $\mathbb{R}^n\backslash\mathcal{S}_{i,0}$, and tends to $+\infty$ as $|x|_{\mathcal{S}_{i,0}} \to +\infty$, there exists a K_∞ function $\underline{\alpha}_{i,1}$ such that, for all $t \in J(x_0, d)$,

$$\begin{aligned}|x(t)|_{\mathcal{S}_{i,0}} \leq \max\{&\underline{\alpha}_{i,1}^{-1}(\Delta_{i,0}),\\ &\underline{\alpha}_{i,1}^{-1}(-\hat{\phi}_{i,1}(-\Delta_{3-i,0})), \underline{\alpha}_{i,1}^{-1}(-\eta_{i,0}^*)\}, \quad i = 1, 2.\end{aligned}$$

Thus, the distance from $x(t)$ to $\mathcal{X} = \mathcal{S}_{1,0} \bigcap \mathcal{S}_{2,0}$ is finite. Together with the assumption that the distance between $x(t)$ and $\mathbb{R}^n\backslash\mathcal{X}$ is bounded, the solution $x(t)$ is well-defined for all $t \in I(x_0, d) = J(x_0, d)$.

B. Proof of (ii) of Theorem 2

Due to the existence of solutions on $I(x_0, d) = J(x_0, d)$, $\inf_{t\in I(x_0,d)} \eta_{i,k}(t)$ is well-defined for $i = 1, 2$ and $k = 1, \ldots, r$. By applying Lemma 3 to (21), with taking $\eta = \eta_{i,k-1}$, $\alpha = \alpha_{i,k}$, and $w = \eta_{i,k}$, there exists an EKL function $\rho_{i,k}$ satisfying $\rho_{i,k}(s, 0) = s$ such that

$$\begin{aligned}\eta_{i,k-1}(t) &\geq \rho_{i,k}(\eta_{i,k-1}(0) - \alpha_{i,k}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,k}(t)), t) + \alpha_{i,k}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,k}(t))\\ &\geq \min\{\eta_{i,k-1}(0), \alpha_{i,k}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,k}(t))\}\\ &\geq \min\{\eta_{i,k-1}^*, \alpha_{i,k}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,k}(t))\}, \quad \forall x_0 \in C, \ \forall t \in I(x_0, d)\end{aligned}$$

with $\eta_{i,k-1}^*$ given in (23). Because $\eta_{i,0}^* = \alpha_{i,1}^{-1} \circ \cdots \circ \alpha_{i,k}^{-1}(\eta_{i,k}^*)$ for each $k = 1, \ldots, r$, we have

$$\begin{aligned}h_i(t) &= \eta_{i,0}(t)\\ &\geq \min\{\eta_{i,0}^*, \alpha_{i,1}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,1}(t))\}\\ &\geq \min\{\eta_{i,0}^*, \alpha_{1,1}^{-1}(\eta_{1,1}^*), \alpha_{i,1}^{-1} \circ \alpha_{i,2}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,2}(t))\}\\ &= \min\{\eta_{i,0}^*, \alpha_{i,1}^{-1} \circ \alpha_{i,2}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,2}(t))\}\\ &\cdots\\ &\geq \min\{\eta_{i,0}^*, \alpha_{i,1}^{-1} \circ \cdots \circ \alpha_{i,r}^{-1}(\inf_{t\in I(x_0,d)} \eta_{i,r}(t))\}\end{aligned}$$

By combining this with (22) and (50),

$$\begin{aligned}h_i(t) &\geq \min\{\eta_{i,0}^*, \hat{\phi}_{i,1}(\inf_{t\in I(x_0,d)} h_{3-i}(t)), -\hat{\gamma}_{i,1}(\|d\|)\}\\ &= \min\{\eta_{i,0}^*, \hat{\phi}_{i,1}(\inf_{t\in I(x_0,d)} h_{3-i}(t))\}, \quad \forall x_0 \in C, \ \forall t \in I(x_0, d).\end{aligned}$$

Therefore,

$$\begin{aligned}\inf_{t\in I(x_0,d)} h_2(t) &\geq \min\{\eta_{2,0}^*, \hat{\phi}_{2,1}(\inf_{t\in I(x_0,d)} h_1(t))\}\\ &\geq \min\{\eta_{2,0}^*, \hat{\phi}_{2,1}(\eta_{1,0}^*), \hat{\phi}_{2,1} \circ \hat{\phi}_{1,1}(\inf_{t\in I(x_0,d)} h_2(t))\}.\end{aligned}$$

With the small-gain condition (31),

$$\begin{aligned}\hat{\phi}_{2,1}(\eta^*_{1,0}) &= \min\{\hat{\phi}_{2,1}\circ\hat{\phi}_{1,1}(-\hat{\gamma}_{2,1}(\|d\|)), \hat{\phi}_{2,1}(-\hat{\gamma}_{1,1}(\|d\|))\}\\ &\geq \min\{\hat{\phi}_{2,1}(-\gamma_{1,1}(\|d\|)), -\hat{\gamma}_{2,1}(\|d\|)\}\\ &= \eta^*_{2,0}.\end{aligned}$$

Therefore,

$$\inf_{t\in I(x_0,d)} h_2(t) \geq \min\{\eta^*_{2,0}, \hat{\phi}_{2,1}\circ\hat{\phi}_{1,1}(\inf_{t\in I(x_0,d)} h_2(t))\}.$$

If $\inf_{t\in I(x_0,d)} h_2(t) \geq 0$, then

$$\hat{\phi}_{2,1}\circ\hat{\phi}_{1,1}(\inf_{t\in I(x_0,d)} h_2(t)) \geq 0 \geq \eta^*_{2,0},$$

and if $\inf_{t\in I(x_0,d)} h_2(t) \leq 0$, then, with the small-gain condition (31),

$$\hat{\phi}_{2,1}\circ\hat{\phi}_{1,1}(\inf_{t\in I(x_0,d)} h_2(t)) \geq \inf_{t\in I(x_0,d)} h_2(t) \geq \eta^*_{2,0}.$$

In summary,

$$\inf_{t\in I(x_0,d)} h_2(t) \geq \eta^*_{2,0}. \tag{58}$$

Combining (22), (50) and (58),

$$\begin{aligned}&\inf_{t\in I(x_0,d)} \eta_{1,r}(t)\\ &\quad\geq \min\{(\mathrm{Id}+\sigma)\circ\phi_1(\inf_{t\in I(x_0,d)} h_2(t)), (\mathrm{Id}+\sigma^{-1})(-\gamma(\|d\|))\}\\ &\quad\geq \min\{(\mathrm{Id}+\sigma)\circ\phi_1(\eta^*_{2,0}), (\mathrm{Id}+\sigma^{-1})(-\gamma(\|d\|))\}\\ &\quad= \min\{(\mathrm{Id}+\sigma)\circ\phi_1\circ\hat{\phi}_{2,1}(-\hat{\gamma}_{1,1}(\|d\|)),\\ &\qquad(\mathrm{Id}+\sigma)\circ\phi_1(-\hat{\gamma}_{2,1}(\|d\|)), (\mathrm{Id}+\sigma^{-1})(-\gamma_1(\|d\|))\}.\end{aligned}$$

Because

$$\begin{aligned}&(\mathrm{Id}+\sigma)\circ\phi_1\circ\hat{\phi}_{2,1}(-\hat{\gamma}_{1,1}(\|d\|))\\ &\quad= (\mathrm{Id}+\sigma)\circ\phi_1\circ\hat{\phi}_{2,1}\circ\alpha^{-1}_{1,1}\\ &\qquad\circ\cdots\circ\alpha^{-1}_{1,r}\circ(\mathrm{Id}+\sigma^{-1})(-\gamma_1(\|d\|))\\ &\quad= (\mathrm{Id}+\sigma)\circ\phi_1\circ\hat{\phi}_{2,1}\circ\hat{\phi}_{1,1}\\ &\qquad\circ\phi_1^{-1}\circ(\mathrm{Id}+\sigma)^{-1}\circ(\mathrm{Id}+\sigma^{-1})(-\gamma_1(\|d\|))\\ &\quad\geq (\mathrm{Id}+\sigma^{-1})(-\gamma_1(\|d\|)),\end{aligned}$$

we have

$$\eta_{1,r}(x_i(t)) \geq \inf_{t\in I(x_0,d)} \eta_{1,r}(t) \geq \eta^*_{1,r} = -|\eta^*_{1,r}|, \quad \forall t\in I(x_0,d) \tag{59}$$

where

$$\eta^*_{1,r} = \min\{(\mathrm{Id}+\sigma)\circ\phi_1(-\hat{\gamma}_{2,1}(\|d\|)), (\mathrm{Id}+\sigma^{-1})(-\gamma_1(\|d\|))\}.$$

Therefore, if we regard $\eta^*_{1,r}$ as an input, then $h_1(x_1)$ is actually a relative-degree-r ISSf-BF satisfying (21) and (59). According to Theorem 1, the state $x(t)$ does not leave the set $\bigcap_{k=1}^r C_{1,k-1}$ for all $t\in I(x_0,d)$ if $x_0\in C$. By symmetry, $x(t)$ also stays in the set $\bigcap_{k=1}^r C_{2,k-1}$ for all $t\in I(x_0,d)$. Thus, $C=\bigcap_{i=1,2}\bigcap_{k=1}^r C_{i,k-1}$ is robustly forward invariant. Now we take $\mathcal{X}_d = C_{1,0}\bigcap C_{2,0}$. Clearly, $\mathcal{X}_d$ is a larger set containing the set $\mathcal{X}$ in (20). Because C is a subset of $\mathcal{X}_d$, $x(t)$ always stays inside $\mathcal{X}_d$, and thus, system (19) is ISSf on $\mathcal{X}$.

C. Proof of (iii) of Theorem 2

From (i) of Theorem 2, $J(x_0,d)=\mathbb{R}_{\geq 0}$ implies that system (19) is forward complete.

We first show that the interconnected system (19) is uniformly stable with respect to the set C. Let

$$\tilde{V}_{i,k-1}(x_i) = \max\{0, -\eta_{i,k-1}(x_i)+\eta^*_{i,k-1}\} \tag{60}$$

for $i=1,2$ and $k=1,\dots,r$. Then, $\tilde{V}_{i,k-1}(x_i)=0$ if $x\in C_{i,k-1}$, and $\tilde{V}_{i,k-1}(x_i)>0$ if $x\in\mathbb{R}^n\backslash C_{i,k-1}$. By combining (51), (56) and (60),

$$\begin{aligned}\|\tilde{V}_{1,0}\| &= \max\{0, \sup_{t\geq 0} -\eta_{1,0}(t)+\eta^*_{1,0}\}\\ &\leq \max\{0, \|V_{1,0}\|+\eta^*_{1,0}\}\\ &\leq \max\{0, \Delta_{1,0}+\eta^*_{1,0}, -\hat{\phi}_{1,1}(-\Delta_{2,0})+\eta^*_{1,0}\}\\ &\leq \max\{0, \mu_{1,1}(\tilde{V}_{1,0}(0)),\dots,\mu_{1,r-1}(\tilde{V}_{1,r-1}(0)),\\ &\qquad\mu_{2,1}(\tilde{V}_{2,0}(0)),\dots,\mu_{2,r-1}(\tilde{V}_{2,r-1}(0))\}\end{aligned} \tag{61}$$

where

$$\begin{aligned}\mu_{1,1}(s) &= s,\\ \mu_{1,k}(s) &= -\alpha^{-1}_{1,1}\circ\cdots\circ\alpha^{-1}_{i,k-1}(-s+\eta^*_{1,k-1})+\eta^*_{1,0}\\ \mu_{2,1}(s) &= -\hat{\phi}_{1,1}(-s+\eta^*_{2,0})+\eta^*_{1,0},\\ \mu_{2,k}(s) &= -\hat{\phi}_{1,1}\circ\hat{\alpha}^{-1}_{2,1}\circ\cdots\circ\alpha^{-1}_{2,k-1}(-s+\eta^*_{2,k-1})+\eta^*_{1,0}\end{aligned}$$

for $k=2,\dots,r$. Clearly, $\mu_{i,k}(s)$ is strictly increasing and tends to $+\infty$ as $s\to+\infty$. Also, it is easy to see that $\mu_{1,1}$ is a K_∞ function. For each $k=2,\dots,r$, it follows from (23) that $\mu_{1,k}(0)=-\alpha^{-1}_{1,1}\circ\cdots\circ\alpha^{-1}_{i,k-1}(\eta^*_{i,k-1})+\eta^*_{1,0}=0$, and hence, $\mu_{1,k}$ is of class K_∞. On the other hand, with the small-gain condition (31),

$$\begin{aligned}\mu_{2,1}(0) &= -\hat{\phi}_{1,1}(\eta^*_{2,0})+\eta^*_{1,0},\\ &= -\min\{\hat{\phi}_{1,1}\circ\hat{\phi}_{2,1}(-\hat{\gamma}_{1,1}(\|d\|)), \hat{\phi}_{1,1}(-\hat{\gamma}_{2,1}(\|d\|))\}+\eta^*_{1,0}\\ &\leq -\min\{-\hat{\gamma}_{1,1}(\|d\|), \hat{\phi}_{1,1}(-\hat{\gamma}_{2,1}(\|d\|))\}+\eta^*_{1,0}\\ &= 0.\end{aligned}$$

Similarly, we also have $\mu_{2,k}(0)\leq 0$ for $k=2,\dots,r$. Therefore, we can assume that $\mu_{2,1},\dots,\mu_{2,r}$ are K_∞ function. Otherwise, we can replace them by

$$\hat{\mu}_{2,k-1}(s) = \mu_{2,k}(s)-\mu_{2,k}(0), \quad k=1,\dots,r.$$

Let

$$\tilde{V}(x) = \max_{i=1,2}\max_{k=1,\dots,r}\tilde{V}_{i,k-1}(x_i) \tag{62}$$

and

$$\delta_{1,1}(s) = \max_{i=1,2}\max_{k=1,\dots,r}\mu_{i,k}(s).$$

Clearly, $\delta_{1,1}$ is of class K_∞. By combining (61) and (62),

$$\begin{aligned}\tilde{V}_{1,0}(x_1(t)) &\leq \|\tilde{V}_{1,0}\|\\ &\leq \max\{\mu_{1,1}(\tilde{V}(x_0)),\dots,\mu_{1,r-1}(\tilde{V}(x_0)),\\ &\qquad\mu_{2,1}(\tilde{V}(x_0)),\dots,\mu_{2,r-1}(\tilde{V}(x_0))\}\\ &= \delta_{1,1}(\tilde{V}(x_0)), \quad \forall x_0\in\mathbb{R}^n,\ \forall t\geq 0.\end{aligned}$$

Similarly, we can construct $\delta_{i,k} \in K_\infty$ for each $i = 1, 2$ and $k = 1, \dots, r$ such that

$$\tilde{V}_{i,k-1}(x_i(t)) \le \delta_{i,k}(\tilde{V}(x_0)), \quad \forall t \ge 0.$$

Thus,

$$\tilde{V}(x(t)) \le \delta(\tilde{V}(x_0)), \quad \forall t \ge 0$$

with $\delta \in K_\infty$ given by

$$\delta(s) = \max_{i=1,2} \max_{k=1,\dots,r} \delta_{i,k}(s).$$

Because $\tilde{V}$ is zero inside C, positive outside C, and tends to $+\infty$ as $|x|_C \to +\infty$, there are $\underline{\mu}, \bar{\mu} \in K_\infty$ such that

$$\underline{\mu}(|x|_C) \le \tilde{V}(x) \le \bar{\mu}(|x|_C), \tag{63}$$

and thus,

$$|x(t)|_C \le \underline{\mu}^{-1} \circ \delta \circ \bar{\mu}(|x_0|_C), \quad \forall x_0 \in \mathbb{R}^n, \ \forall t \ge 0,$$

which, together with (2), implies that the interconnected system (19) is uniformly stable with respect to C.

The rest is to prove that the interconnected system (19) is uniformly attractive with respect to C. Recalling (53), because $\beta_{i,k}(s,t)$ tends to zero as $t \to +\infty$,

$$\limsup_{t\to+\infty} V_{i,k-1}(x_i(t)) \le -\alpha_{i,k}^{-1}(-\sup_{t\ge 0} V_{i,k}(x_i(t)))$$

for $i = 1, 2$ and $k = 1, \dots, r$. According to [47, Lemma 10.4.4],

$$\limsup_{t\to+\infty} V_{i,k-1}(x_i(t)) \le -\alpha_{i,k}^{-1}(-\limsup_{t\to+\infty} V_{i,k}(x_i(t))). \tag{64}$$

Similarly, from (55)

$$\begin{aligned} \limsup_{t\to+\infty} V_{i,r}(x_i(t)) \le \max\{&-(\mathrm{Id} + \sigma^{-1})(-\gamma_i(\|d\|)), \\ &-(\mathrm{Id} + \sigma) \circ \phi_i(-\limsup_{t\to+\infty} V_{3-i,0}(x_{3-i}(t)))\}. \end{aligned} \tag{65}$$

By combining (64) and (65),

$$\begin{aligned} &\limsup_{t\to+\infty} V_{i,0}(x_i(t)) \\ &\quad \le \limsup_{t\to+\infty} -\alpha_{i,1}^{-1} \circ \cdots \circ \alpha_{i,r}^{-1}(-V_{i,r}(x_i(t))) \\ &\quad \le \max\Big\{ -\hat{\phi}_{i,1}(-\limsup_{t\to+\infty} V_{3-i,0}(-x_{3-i}(t))), \hat{\gamma}_{i,1}(\|d\|)\Big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\limsup_{t\to+\infty} V_{1,0}(x_1(t)) \\ &\quad \le \max\Big\{ -\hat{\phi}_{1,1} \circ \hat{\phi}_{2,1}(-\limsup_{t\to+\infty} V_{1,0}(x_{1,0}(t))), -\eta_{1,0}^*\Big\}, \\ &\quad \le -\eta_{1,0}^*, \end{aligned}$$

where the final inequality follows from the small-gain condition (31). With a similar derivation,

$$\limsup_{t\to+\infty} V_{i,k-1}(x_i(t)) \le -\eta_{i,k-1}^*, \quad i = 1, 2, \ k = 1, \dots, r.$$

Therefore,

$$\begin{aligned} &\limsup_{t\to+\infty} \tilde{V}_{i,k-1}(x_i(t)) \\ &\qquad = \max\{0, \limsup_{t\to+\infty} -\eta_{i,k-1}(x_i(t)) + \eta_{i,k-1}^*\} \\ &\qquad \le \max\{0, \limsup_{t\to+\infty} V_{i,k-1}(x_i(t)) + \eta_{i,k-1}^*\} \\ &\qquad \le 0, \end{aligned}$$

and consequently,

$$\limsup_{t\to+\infty} \tilde{V}(x(t)) = \limsup_{t\to+\infty} \max_{i=1,2} \max_{k=1,\dots,r} \tilde{V}_{i,k-1}(x_i(t)) \le 0.$$

Recalling (63), we have

$$0 \le \limsup_{t\to+\infty} |x(t)|_C \le \underline{\mu}^{-1}(\limsup_{t\to+\infty} \tilde{V}(x(t))) \le 0,$$

which implies that the interconnected system (19) is uniformly attractive with respect to C.

In summary, the interconnected system (19) is UGAS with respect to the set C.

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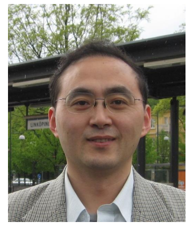

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