Data-driven Stability Analysis of Neural Network Control Systems

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***Abstract*— This letter studies the problem of verifying stability of neural network control systems in which the plant dynamics are unknown but the neural network controller is known. We derive sufficient conditions for stability that are directly computable from a finite-length trajectory of the closed-loop system, without requiring an explicit model of the plant. The proposed conditions take the form of linear matrix inequalities and, as a byproduct, yield an estimate of the region of attraction. Both the noiseless and noisy data settings are considered: in the noiseless case, data-driven sufficient conditions are derived to guarantee stability; in the noisy case, sufficient conditions are derived to ensure robust stability of all systems consistent with the data. The effectiveness of the proposed approach is demonstrated on an inverted pendulum example.**

I. Introduction

Neural network control systems (NNCSs) – feedback systems in which neural networks (NNs) appear in the control loop – have attracted growing interest in recent years. Such NNs arise in various practical contexts including vision-based perception modules [1], compact approximations of large lookup tables [2], controllers synthesized via reinforcement learning (RL) [3], or surrogates trained to replicate computationally expensive model predictive control (MPC) policies [4], [5].

Verifying the stability of NNCSs has been investigated through a variety of methods centered on constructing candidate Lyapunov functions, including optimization-based methods [6], [7], [8] and self-supervised learning approaches [9]. A particularly prominent line of work abstracts the nonlinear activation functions of NNs using quadratic constraints (QCs), enabling stability verification and robustness analysis via semidefinite programming [10], [11], [12], [13]. Despite these advances, existing QC-based approaches generally rely on explicit knowledge of the plant dynamics, which limits their applicability when accurate system models are unavailable.

Obtaining system models from first principles is often prohibitively costly, and system identification introduces additional layers of error and uncertainty that are difficult to quantify and propagate into downstream stability certificates [14]. Data-driven control has emerged as a compelling alternative, enabling the design and analysis of control systems directly from time-series data without requiring an

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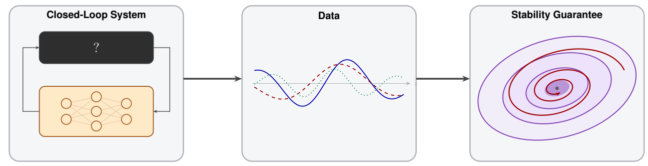

Fig. 1. Data-driven stability analysis of an unknown LTI system in feedback with a known feedforward neural network controller.

explicit model. Particularly relevant to the present work are results rooted in the behavioral approach [15] and the data informativity framework [16], which characterizes the set of all systems consistent with observed data and establishes system-theoretic properties such as dissipativity and stability [17], [18], [19]. These results, however, are confined to conventional linear time-invariant systems and do not extend to the nonlinear setting inherent to NNCSs.

We close this gap by proposing a data-driven, QC-based approach for the stability analysis of NNCSs in which the plant model is unknown while the NN controller is known (see Fig. 1). Such a setting naturally arises when a pre-trained NN controller is deployed on a plant whose dynamics are not explicitly modeled and can only be accessed through measured data; in these situations, formal guarantees of closed-loop stability are often unavailable. Using input-state data collected from the system, we derive sufficient conditions, formulated as linear matrix inequalities (LMIs), for certifying closed-loop stability and computing inner-approximations of the region of attraction (ROA). Separate results are established for the cases of noiseless data (Theorem 1) and data corrupted by noise (Theorem 2). The remainder of this letter is organized as follows. Section II presents the problem statement and reviews nominal stability analysis for the case of a known plant. Section III develops data-driven stability conditions for noiseless and noisy data separately. Section IV presents a numerical example and Section V provides concluding remarks.

Notation. The $n \times n$ identity matrix is denoted as I_n. The $n \times m$ matrix whose entries are all 0 is denoted as $\mathbf{0}_{n\times m}$, where the dimension subscripts may be omitted when clear from context. The set of non-negative vectors is denoted as $\mathbb{R}^n_{\geq 0}$. The sets of $n \times n$ symmetric matrices, positive semi-definite matrices, and positive definite matrices in the real space are denoted as $\mathbb{S}^n$, $\mathbb{S}^n_{\succeq 0}$, and $\mathbb{S}^n_{\succ 0}$, respectively. For symmetric matrices, $*$ denotes entries whose values follow from symmetry. For a given square matrix M, $M \succ 0, M \succeq 0, M \prec 0, M \preceq 0$ mean M is positive definite, positive semidefinite, negative definite, and negative semidefinite, respectively. For a sequence of matrices $A_1, \cdots A_n$, $\mathrm{diag}(A_1, \cdots, A_n)$ denotes the block diagonal matrix with diagonal blocks $A_1, \cdots, A_n$ in that order. For a positive integer n, denote $[n] = \{1, 2, \cdots, n\}$.

II. Problem Statement & Preliminaries

A. Problem Statement

Consider the discrete-time linear system

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are unknown. The controller $u(t)$ is given as

$$u(t) = \pi(x(t)), \tag{2}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is a known, ℓ-layer, feedforward neural network (FNN) defined as follows:

$$\begin{aligned} &w^0(t) = x(t), \\ &w^i(t) = \phi^i\left(W^i w^{i-1}(t) + b^i\right), \ \forall i \in [\ell], \\ &u(t) = W^{\ell+1} w^\ell(t) + b^{\ell+1}. \end{aligned} \tag{3}$$

Here, $W^i \in \mathbb{R}^{n_i \times n_{i-1}}$, $b^i \in \mathbb{R}^{n_i}$, and $w^i \in \mathbb{R}^{n_i}$ are the weight matrix, bias vector, and output (activation) of the i^{th} ($i \in [\ell]$) layer, respectively, with $n_0 = n$ and $n_{\ell+1} = m$; $\phi^i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is a vector activation function defined as

$$\phi^i(v) \triangleq \left[\varphi(v_1), \cdots, \varphi(v_{n_i})\right]^\top, \tag{4}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a scalar activation function (e.g., ReLU, sigmoid). Although in this work we assume that φ is identical across all layers, the results can be readily extended to the case where different activation functions are used in different layers with minor notational changes, provided that all activation functions involved satisfy the sector-bounded property and can be abstracted using QCs, which will be described in the next section. We make the following assumption on the FNN π.

Assumption 1: The FNN π satisfies $\pi(\mathbf{0}) = \mathbf{0}$ and $W^{\ell+1} = I_m$.

Remark 1: For any FNN, an additional output layer with an identity weight matrix can be appended without altering its input-output mapping. In the QC formulation, this identity layer can be represented as a degenerate sector-bounded relation by setting the sector bounds $\alpha = \beta = 1$, which enforces the identity mapping exactly. Although this modification does not change the controller itself, it increases the dimension of the QC characterization and may introduce a small amount of additional conservatism in the subsequent QC-based analysis. In practice, however, this assumption is not restrictive as many NN controllers include an output saturation of the form $u(t) = sat_{\underline{u}}^{\overline{u}}(\pi(x))$ where $\underline{u} \leq \overline{u}$. Since the saturation function can be represented exactly as $sat_{\underline{u}}^{\overline{u}}(u) = \min\{\max\{u, \underline{u}\}, \overline{u}\} = \mathrm{ReLU}(-\mathrm{ReLU}(u - \overline{u}) + u - \underline{u}) + \underline{u}$, the resulting saturated NN contains a final layer with an identity weight matrix, which makes the identity assumption automatically satisfied.

The NNCS consisting of the system (1) and controller (2) is a closed-loop system

$$x(t+1) = Ax(t) + B\pi(x(t)). \tag{5}$$

Because $\pi(\mathbf{0}) = \mathbf{0}$, the origin $x_* = \mathbf{0}$ is an equilibrium point of the NNCS (5).

In this letter, we investigate conditions that certify the local stability of the closed-loop system (5), whose matrices A and B are *unknown*, using a finite set of input-state data $\{u(t)\}_{t=0}^{T-1}$ and $\{x(t)\}_{t=0}^{T}$. The data are assumed to be either noiseless, i.e., generated by the system (5), or contaminated by additive noise, in which case they are generated by

$$x(t+1) = Ax(t) + B\pi(x(t)) + e(t), \tag{6}$$

where $e(t) \in \mathbb{R}^n$ denotes unknown noise.

Certifying stability of (5) could be approached by identifying (A, B) from data and applying model-based stability analysis. This indirect approach involves system identification, uncertainty quantification, and robust model-based stability analysis, which can involve significant practical effort for NNCS. In this work, we develop data-based conditions that certify robust stability for every system consistent with the observed data, bypassing system identification entirely. Sufficient conditions in the form of LMIs are derived for the noiseless case in Section III-A and for the noisy case in Section III-B.

B. Stability of NNCS with Known Plant Model

In this subsection, we review the stability analysis result for the NNCS (5) with matrices A and B known [13].

Define v^i as the input to the activation function ϕ^i:

$$v^i(t) \triangleq W^i w^{i-1}(t) + b^i, \ \forall i \in [\ell]. \tag{7}$$

Define the concatenated forms of the inputs $v_\phi \in \mathbb{R}^{n_\phi}$ and outputs $w_\phi \in \mathbb{R}^{n_\phi}$ of all activation functions as

$$v_\phi \triangleq \begin{bmatrix} v^1 \\ \vdots \\ v^\ell \end{bmatrix}, \quad w_\phi \triangleq \begin{bmatrix} w^1 \\ \vdots \\ w^\ell \end{bmatrix}, \tag{8}$$

where $n_\phi \triangleq \sum_{i=1}^{\ell} n_i$ is the total number of neurons in π. Define the combined nonlinearity $\phi : \mathbb{R}^{n_\phi} \to \mathbb{R}^{n_\phi}$ as

$$\phi(v_\phi) \triangleq \begin{bmatrix} \phi^1(v^1) \\ \vdots \\ \phi^\ell(v^\ell) \end{bmatrix}, \tag{9}$$

where the scalar activation function φ in ϕ^i, defined in (4), is applied element-wise to each entry of v_ϕ. It is easy to see that variables $u(t), x(t), v_\phi(t), w_\phi(t)$ are related by

$$\begin{bmatrix} v_\phi(t) \\ u(t) \end{bmatrix} = N \begin{bmatrix} x(t) \\ w_\phi(t) \\ 1 \end{bmatrix}, \tag{10}$$

$$w_\phi(t) = \phi(v_\phi(t)), \tag{11}$$

where

$$N = \left[\begin{array}{c|cccc|c} W^1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & b^1 \\ \mathbf{0} & W^2 & \cdots & \mathbf{0} & \mathbf{0} & b^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & W^\ell & \mathbf{0} & b^\ell \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & W^{\ell+1} & b^{\ell+1} \end{array}\right] \triangleq \left[\begin{array}{c|c|c} N_{vx} & N_{vw} & N_{vb} \\ \hline N_{ux} & N_{uw} & N_{ub} \end{array}\right]. \tag{12}$$

We can obtain the equilibrium values v_*^i, w_*^i ($i \in [\ell]$) by propagating $x_* = \mathbf{0}$ through π, yielding $u_* = \mathbf{0}$ and $v_\phi = v_*, w_\phi = w_*$. By construction, (x_*, u_*, v_*, w_*) are unique and satisfy the following conditions:

$$x_* = Ax_* + Bu_*, \quad \begin{bmatrix} v_* \\ u_* \end{bmatrix} = N \begin{bmatrix} x_* \\ w_* \\ 1 \end{bmatrix}, \quad w_* = \phi(v_*). \tag{13}$$

The nonlinearities of the activation function in the FNN π can be abstracted using the sector-bounded properties of the activation function via QCs. Specifically, let $\alpha, \beta, \underline{v}, \overline{v}, v_0 \in \mathbb{R}$ be given with $\alpha \leq \beta$ and $\underline{v} \leq v_0 \leq \overline{v}$; the function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies the offset local sector $[\alpha, \beta]$ around the point $(v_0, \varphi(v_0))$, if $[\varphi(v) - \varphi(v_0) - \alpha(v - v_0)] \cdot [\varphi(v) - \varphi(v_0) - \beta(v - v_0)] \leq 0, \forall v \in [\underline{v}, \overline{v}]$ [13, Def. 4]. The sectors can be stacked into vectors $\alpha_\phi, \beta_\phi \in \mathbb{R}^{n_\phi}$ that provide QCs satisfied by the combined nonlinearity ϕ defined in (9).

Assumption 2: Let $\alpha_\phi, \beta_\phi, \underline{v}, \overline{v}, v_*, w_* \in \mathbb{R}^{n_\phi}$ be given vectors such that $\alpha_\phi \leq \beta_\phi$, $\underline{v} \leq v_* \leq \overline{v}$, and (13) holds. The nonlinearity ϕ satisfies the local sector $[\alpha_\phi, \beta_\phi]$ around (v_*, w_*), entry-wise for all $v_\phi \in [\underline{v}, \overline{v}]$.

The bounds $\underline{v}, \overline{v} \in \mathbb{R}^{n_\phi}$ in Assumption 2 can be computed layer by layer through the network; see Section III.D of [13].

Lemma 1: [13, Lemma 1] Suppose that Assumptions 1 and 2 hold. For any $\lambda \in \mathbb{R}^{n_\phi}_{\geq 0}$ and $v_\phi \in [\underline{v}, \overline{v}]$, it holds that

$$\begin{bmatrix} v_\phi - v_* \\ w_\phi - w_* \end{bmatrix}^\top \Psi_\phi^\top M_\phi(\lambda) \Psi_\phi \begin{bmatrix} v_\phi - v_* \\ w_\phi - w_* \end{bmatrix} \geq 0,$$

where $w_\phi = \phi(v_\phi)$ and

$$\Psi_\phi = \begin{bmatrix} \operatorname{diag}(\beta_\phi) & -I_{n_\phi} \\ -\operatorname{diag}(\alpha_\phi) & I_{n_\phi} \end{bmatrix}, \tag{14}$$

$$M_\phi(\lambda) = \begin{bmatrix} \mathbf{0}_{n_\phi} & \operatorname{diag}(\lambda) \\ \operatorname{diag}(\lambda) & \mathbf{0}_{n_\phi} \end{bmatrix}. \tag{15}$$

The following lemma provides a sufficient condition for the asymptotic stability of NNCS (5) with known A and B.

Lemma 2: [13, Theorem 1] Consider the NNCS (5) where matrices A, B are known. Suppose that Assumptions 1 and 2 hold. If there exist a matrix $P \in \mathbb{S}^n_{\succ 0}$ and a vector $\lambda \in \mathbb{R}^{n_\phi}_{\geq 0}$ such that

$$R_V^\top \left[\begin{array}{cc} A^\top PA - P & A^\top PB \\ * & B^\top PB \end{array} \right] R_V + X(\lambda) \prec 0, \tag{16}$$

$$\left[\begin{array}{cc} \left(\overline{v}_i^1 - v_{*,i}^1\right)^2 & W_i^1 \\ * & P \end{array} \right] \succeq 0, \quad \forall i \in [n_1], \tag{17}$$

where W_i^1 is the i-th row of W^1, $v_{*,i}^1$ is the i-th entry of v_*^1,

$$X(\lambda) = R_\phi^\top \Psi_\phi^\top M_\phi(\lambda) \Psi_\phi R_\phi, \tag{18}$$

with

$$R_V = \begin{bmatrix} I_n & \mathbf{0}_{n \times n_\phi} \\ N_{ux} & N_{uw} \end{bmatrix}, \quad R_\phi = \begin{bmatrix} N_{vx} & N_{vw} \\ \mathbf{0}_{n_\phi \times n} & I_{n_\phi} \end{bmatrix}, \tag{19}$$

then the NNCS (5) is locally asymptotically stable around the origin, and the set $\mathcal{E}(P, \mathbf{0}_n) \triangleq \{x \in \mathbb{R}^n | x^\top Px \leq 1\}$ is an inner-approximation of the ROA.

We denote the left-hand side of (16) as $\Omega(P, \lambda)$, i.e.,

$$\Omega(P, \lambda) \triangleq R_V^\top \left[\begin{array}{cc} A^\top PA - P & A^\top PB \\ * & B^\top PB \end{array} \right] R_V + X(\lambda), \tag{20}$$

which will be used in the subsequent sections.

III. Main Results

A. Data-driven Stability with Noiseless Data

In this subsection, data-driven stability conditions are developed for the NNCS (5) using a finite set of noiseless input-state data $\{u(t)\}_{t=0}^{T-1}$ and $\{x(t)\}_{t=0}^{T}$ generated by (5). While the result is of independent theoretical interest, it primarily serves as the theoretical foundation for the noisy data case addressed in the next subsection.

We sort the state and input data into matrices

$$U_- \triangleq [u(0), u(1), \ldots, u(T-1)] \in \mathbb{R}^{m \times T}, \tag{21a}$$

$$X_- \triangleq [x(0), x(1), \ldots, x(T-1)] \in \mathbb{R}^{n \times T}, \tag{21b}$$

$$X_+ \triangleq [x(1), x(2), \ldots, x(T)] \in \mathbb{R}^{n \times T}, \tag{21c}$$

which satisfy the equation $X_+ = AX_- + BU_-$.

The following theorem provides sufficient LMI-based conditions for the local asymptotic stability of the NNCS (5) with an unknown plant model, based on the noiseless data matrices defined in (21).

Theorem 1: Consider the NNCS (5) where the plant matrices A, B are unknown. Suppose that Assumptions 1 and 2 hold. Assume that the data $\{u(t)\}_{t=0}^{T-1}$ and $\{x(t)\}_{t=0}^{T}$ are generated by (5) where $T = m + n$ and the data matrices X_-, U_- defined in (21) satisfy

$$\operatorname{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = m + n. \tag{22}$$

If there exist $P \in \mathbb{S}^n_{\succ 0}$, $\lambda \in \mathbb{R}^{n_\phi}_{\geq 0}$ such that (17) holds and

$$Z_1(P) + Z_2(\lambda) \prec 0, \tag{23}$$

where

$$Z_1(P) = \begin{bmatrix} X_- & \mathbf{0}_{n \times \bar{n}_\phi} \\ X_+ & \mathbf{0}_{n \times \bar{n}_\phi} \end{bmatrix}^\top \begin{bmatrix} -P & \mathbf{0}_n \\ \mathbf{0}_n & P \end{bmatrix} \begin{bmatrix} X_- & \mathbf{0}_{n \times \bar{n}_\phi} \\ X_+ & \mathbf{0}_{n \times \bar{n}_\phi} \end{bmatrix}, \tag{24a}$$

$$Z_2(\lambda) = \begin{bmatrix} X_- & \mathbf{0}_{n \times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m \times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}^\top \tilde{R}_\phi^\top \Psi_\phi^\top M_\phi(\lambda) \Psi_\phi \tilde{R}_\phi \cdot \begin{bmatrix} X_- & \mathbf{0}_{n \times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m \times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}, \tag{24b}$$

$$\tilde{R}_\phi = \begin{bmatrix} N_{vx} & \tilde{N}_{vw} \\ \mathbf{0}_{n_\phi \times n} & \tilde{I}_{n_\phi} \end{bmatrix} \triangleq \left[\begin{array}{c|cccc} W^1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & W^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & W^\ell \\ \hline \mathbf{0} & \mathbf{0} & & I_{\bar{n}_\phi} & \\ \mathbf{0} & I_m & & \mathbf{0} & \end{array} \right], \tag{24c}$$

Ψ_ϕ and $M_\phi(\lambda)$ are defined in (14) and (15), respectively, and $\bar{n}_\phi = n_\phi - m$, then the NNCS (5) is locally asymptotically stable around the origin, and the set $\mathcal{E}(P, \mathbf{0}_n) \triangleq \{x \in \mathbb{R}^n | x^\top Px \leq 1\}$ is an inner-approximation of the ROA.

Proof: Since $X_+ = AX_- + BU_-$, we have

$$Z_1(P) + Z_2(\lambda)$$

$$= \begin{bmatrix} X_- & \mathbf{0}_{n \times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m \times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}^\top \begin{bmatrix} I_n & \mathbf{0}_{n \times m} & \mathbf{0}_{n \times \bar{n}_\phi} \\ A & B & \mathbf{0}_{n \times \bar{n}_\phi} \end{bmatrix}^\top \begin{bmatrix} -P & \mathbf{0}_n \\ \mathbf{0}_n & P \end{bmatrix}$$

$$\cdot \begin{bmatrix} I_n & \mathbf{0}_{n\times m} & \mathbf{0}_{n\times \bar{n}_\phi} \\ A & B & \mathbf{0}_{n\times \bar{n}_\phi} \end{bmatrix} \begin{bmatrix} X_- & \mathbf{0}_{n\times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m\times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}$$

$$+ \begin{bmatrix} X_- & \mathbf{0}_{n\times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m\times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}^\top \tilde{R}_\phi^\top \Psi_\phi^\top M_\phi(\lambda) \Psi_\phi \tilde{R}_\phi \begin{bmatrix} X_- & \mathbf{0}_{n\times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m\times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}$$

$$= \begin{bmatrix} X_- & \mathbf{0}_{n\times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m\times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}^\top \begin{bmatrix} I_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_\phi - m} \\ \mathbf{0} & I_m & \mathbf{0} \end{bmatrix}^\top \Omega(P,\lambda)$$

$$\cdot \begin{bmatrix} I_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_\phi - m} \\ \mathbf{0} & I_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} X_- & \mathbf{0}_{n\times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m\times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}.$$

Since $T = n + m$ and condition (22) holds, the matrix $\begin{bmatrix} X_- & \mathbf{0}_{n\times \bar{n}_\phi} \\ U_- & \mathbf{0}_{m\times \bar{n}_\phi} \\ \mathbf{0}_{\bar{n}_\phi \times T} & I_{\bar{n}_\phi} \end{bmatrix}$ is square and non-singular. Moreover, since $\begin{bmatrix} I_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n_\phi - m} \\ \mathbf{0} & I_m & \mathbf{0} \end{bmatrix}$ is also square and non-singular, $Z_1(P) + Z_2(\lambda) \prec 0$ is equivalent to $\Omega(P,\lambda) \prec 0$, that is, (16) holds. Then the conclusion follows by Lemma 2. ∎

Remark 2: The LMI (23) has size $(n + n_\phi) \times (n + n_\phi)$, which coincides with the LMI size in Theorem 1 of [13], where sufficient conditions for the stability of NNCS are developed assuming known system matrices A and B. Moreover, the decision variables in both LMIs consist of a positive definite matrix $P \in \mathbb{S}^n_{\succ 0}$ and a vector $\lambda \in \mathbb{R}^{n_\phi}$. Therefore, the proposed data-driven conditions generalize existing model-based stability conditions for NNCS with the same computational complexity, but without requiring an explicit system identification step. To maximize the inner-approximation of the ROA, one can minimize $\mathrm{trace}(P)$ subject to the LMIs of Theorem 1 as in [13].

Remark 3: Condition (22) ensures that the matrices (A, B) can be uniquely identified [20]. The data length $T = m + n$ is the minimum required for (22) to hold, and such a minimum-length informative trajectory can be obtained from the open-loop system (1) through experiment design [21]. When only closed-loop data from (5) is available, however, the data conditions in Theorem 1 may no longer be satisfied. Indeed, the richness of closed-loop data depends on the dynamics of the interconnected system (i.e., the NN controller and the LTI plant), and on the initial state from which the trajectory is collected. Ensuring sufficient richness of closed-loop data is a central challenge in system identification, and relaxing the data conditions under which stability can be guaranteed is left for future work.

B. Data-driven Stability with Noisy Data

In this subsection, data-driven stability conditions are developed for the NNCS (5) using a finite set of noisy input-state data $\{u(t)\}_{t=0}^{T-1}$ and $\{x(t)\}_{t=0}^{T}$ generated by (6).

Due to the presence of unknown noise $\{e(t)\}_{t=1}^{T}$ in the observed data, the exact system matrices A, B cannot be uniquely determined even under the rank assumption (22). Define the following noise matrix

$$E \triangleq [e(1), e(2), \ldots, e(T)] \in \mathbb{R}^{n\times T}. \tag{25}$$

We assume that the matrix E belongs to the following noise model that characterizes the noise uncertainty:

$$\mathcal{N}_e = \{E \in \mathbb{R}^{n\times T} \mid EQE^\top \prec R\}, \tag{26}$$

where $Q \in \mathbb{S}^T_{\succ 0}$ and $R \in \mathbb{S}^n_{\succ 0}$.

Remark 4: In [18], the following noise model is considered: $\mathcal{N}_1 \triangleq \left\{ E \,\middle|\, \begin{bmatrix} I \\ E^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ E^\top \end{bmatrix} \succeq 0 \right\}$ where $\Phi_{11} = \Phi_{11}^\top$, Φ_{12}, and $\Phi_{22} = \Phi_{22}^\top$ are known matrices. By choosing the submatrices appropriately, this noise model captures a variety of practical noise descriptions, including energy bounds and sample covariance bounds [18], [22]. The noise model (26) corresponds to a special case of $\mathcal{N}_1$ with $\Phi_{11} = R, \Phi_{12} = \mathbf{0}, \Phi_{22} = -Q$.

Inspired by the informativity framework [16], we aim to derive data-based conditions that ensure stability for every system that could have generated the data under the noise model (26). Based on the data matrices X_-, X_+, and U_- defined in (21), we have

$$E = X_+ - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \tag{27}$$

Therefore, the set of all (A, B) consistent with the observed data can be defined as

$$\Sigma_{\mathcal{N}_e} \triangleq \left\{ (A, B) \,\middle|\, X_+ - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X_- \\ U_- \end{bmatrix} \in \mathcal{N}_e \right\}. \tag{28}$$

The following theorem provides sufficient LMI-based conditions for the local stability of the NNCS (5) with an unknown plant model and the noise model in (26), based on the data matrices defined in (21). The key idea of the sufficient conditions is to ensure that the NNCS (5) is asymptotically stable for all $(A, B) \in \Sigma_{\mathcal{N}_e}$.

Theorem 2: Consider the NNCS (5) where matrices A, B are unknown. Suppose that Assumptions 1 and 2 hold. Suppose that the noisy data $\{u(t)\}_{t=0}^{T-1}$ and $\{x(t)\}_{t=0}^{T}$ are generated by (6) where $T = m + n$, the data matrices satisfy (22), and the noise model $\mathcal{N}_e$ is defined in (26). If there exist $P \in \mathbb{S}^n_{\succ 0}$, $\gamma \geq 0$ and $\lambda \in \mathbb{R}^{n_\phi}_{\geq 0}$ such that (17) holds and

$$\begin{bmatrix} Z_1(P) + Z_2(\lambda) & * \\ -\begin{bmatrix} -P & \mathbf{0}_n \\ \mathbf{0}_n & P \end{bmatrix} \begin{bmatrix} X_- & \mathbf{0}_{n\times \bar{n}_\phi} \\ X_+ & \mathbf{0}_{n\times \bar{n}_\phi} \end{bmatrix} & \begin{bmatrix} -P & \mathbf{0}_n \\ \mathbf{0}_n & P \end{bmatrix} \end{bmatrix} - \gamma \begin{bmatrix} -Q^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{\bar{n}_\phi} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & R^{-1} \end{bmatrix} \prec 0, \tag{29}$$

then the NNCS (5) is locally asymptotically stable around the origin, and the set $\mathcal{E}(P, \mathbf{0}_n) \triangleq \{x \in \mathbb{R}^n | x^\top P x \leq 1\}$ is an inner-approximation of the ROA.

Proof: By the dualization lemma [18], we have $\begin{bmatrix} I \\ E^\top \end{bmatrix}^\top \begin{bmatrix} -R & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix} \begin{bmatrix} I \\ E^\top \end{bmatrix} \prec \mathbf{0} \Leftrightarrow \begin{bmatrix} -E \\ I \end{bmatrix}^\top \begin{bmatrix} -R^{-1} & \mathbf{0} \\ \mathbf{0} & Q^{-1} \end{bmatrix} \begin{bmatrix} -E \\ I \end{bmatrix} \succ \mathbf{0} \Leftrightarrow E^\top R^{-1} E \prec Q^{-1}$. Therefore, the noise model (26) can be rewritten as $\mathcal{N}_e = \{E \in \mathbb{R}^{n\times T} \mid \begin{bmatrix} I \\ E^\top \end{bmatrix}^\top \begin{bmatrix} -R & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix} \begin{bmatrix} I \\ E^\top \end{bmatrix} \prec \mathbf{0}\} = \{E \in \mathbb{R}^{n\times T} \mid E^\top R^{-1} E \prec Q^{-1}\}$.

We will now show that (16) is satisfied for all $(A,B)\in\Sigma_{\mathcal{N}_e}$. From LMI (29), any $\xi\in\mathbb{R}^{n_\phi+3n}\backslash\{\mathbf{0}\}$ satisfying $\xi^\top\begin{bmatrix}-Q^{-1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}_{\bar{n}_\phi}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&I_n&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&R^{-1}\end{bmatrix}\xi\le 0$ results in

$$\xi^\top\begin{bmatrix}Z_1(P)+Z_2(\lambda) & *\\ -\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix} & \begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\end{bmatrix}\xi<0.$$

Let $\eta=\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}\in\mathbb{R}^{n_\phi+n}\backslash\{\mathbf{0}\}$ be an arbitrary vector where $\eta_1\in\mathbb{R}^T,\eta_2\in\mathbb{R}^{\bar{n}_\phi}$, and $E\in\mathcal{N}_e$ be an arbitrary noise matrix. Define $\xi\triangleq\begin{bmatrix}\eta\\ \begin{bmatrix}\mathbf{0}_{n\times T}&\mathbf{0}_{n\times\bar{n}_\phi}\\ E&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}\eta\end{bmatrix}=\begin{bmatrix}\eta_1\\ \eta_2\\ \mathbf{0}_n\\ E\eta_1\end{bmatrix}$.

Then, $\xi^\top\begin{bmatrix}-Q^{-1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}_{\bar{n}_\phi}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&I_n&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&R^{-1}\end{bmatrix}\xi=\eta_1^\top(E^\top R^{-1}E-Q^{-1})\eta_1\le 0$, which implies that

$$\begin{aligned}&\xi^\top\begin{bmatrix}Z_1(P)+Z_2(\lambda) & *\\ -\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix} & \begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\end{bmatrix}\xi\\ =&\eta^\top(Z_1(P)+Z_2(\lambda))\eta\\ &-\eta^\top\begin{bmatrix}\mathbf{0}_{n\times T}&\mathbf{0}_{n\times T}\\ E&\mathbf{0}_{n\times T}\end{bmatrix}^\top\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}\eta\\ &-\eta^\top\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}^\top\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}\mathbf{0}_{n\times T}&\mathbf{0}_{n\times T}\\ E&\mathbf{0}_{n\times T}\end{bmatrix}\eta\\ &+\eta^\top\begin{bmatrix}\mathbf{0}_{n\times T}&\mathbf{0}_{n\times T}\\ E&\mathbf{0}_{n\times T}\end{bmatrix}^\top\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}\mathbf{0}_{n\times T}&\mathbf{0}_{n\times T}\\ E&\mathbf{0}_{n\times T}\end{bmatrix}\eta\\ =&\eta^\top(\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+-E&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}^\top\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+-E&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}\\ &+Z_2(\lambda))\eta<0.\end{aligned}$$

Since η in the above inequality is arbitrary, it holds that

$$\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+-E&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}^\top\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}_{n\times\bar{n}_\phi}\\ X_+-E&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}+Z_2(\lambda)\prec 0, \tag{30}$$

for all $E\in\mathcal{N}_e$. From (27), inequality (30) implies that, for any $(A,B)\in\Sigma_{\mathcal{N}_e}$,

$$\begin{aligned}&\begin{bmatrix}X_-&\mathbf{0}\\ \begin{bmatrix}A&B\end{bmatrix}\begin{bmatrix}X_-\\ U_-\end{bmatrix}&\mathbf{0}\end{bmatrix}^\top\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}\\ \begin{bmatrix}A&B\end{bmatrix}\begin{bmatrix}X_-\\ U_-\end{bmatrix}&\mathbf{0}\end{bmatrix}\\ &+Z_2(\lambda)\\ =&\begin{bmatrix}X_-&\mathbf{0}\\ U_-&\mathbf{0}\\ \mathbf{0}&I_{\bar{n}_\phi}\end{bmatrix}^\top\begin{bmatrix}I_n&\mathbf{0}&\mathbf{0}\\ A&B&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}^\top\begin{bmatrix}-P&\mathbf{0}_n\\ \mathbf{0}_n&P\end{bmatrix}\\ &\cdot\begin{bmatrix}I_n&\mathbf{0}&\mathbf{0}\\ A&B&\mathbf{0}_{n\times\bar{n}_\phi}\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}\\ U_-&\mathbf{0}\\ \mathbf{0}&I_{\bar{n}_\phi}\end{bmatrix}+Z_2(\lambda)\\ =&\begin{bmatrix}X_-&\mathbf{0}\\ U_-&\mathbf{0}\\ \mathbf{0}&I_{\bar{n}_\phi}\end{bmatrix}^\top\begin{bmatrix}I_n&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&I_{n_\phi-m}\\ \mathbf{0}&I_m&\mathbf{0}\end{bmatrix}^\top\Omega(P,\lambda)\\ &\cdot\begin{bmatrix}I_n&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&I_{n_\phi-m}\\ \mathbf{0}&I_m&\mathbf{0}\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}\\ U_-&\mathbf{0}\\ \mathbf{0}&I_{\bar{n}_\phi}\end{bmatrix}\prec 0.\end{aligned}$$

Because $\begin{bmatrix}I_n&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&I_{n_\phi-m}\\ \mathbf{0}&I_m&\mathbf{0}\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}\\ U_-&\mathbf{0}\\ \mathbf{0}&I_{\bar{n}_\phi}\end{bmatrix}$ is square and non-singular by the rank condition in (22), we have $\Omega(P,\lambda)\prec 0$ for any $(A,B)\in\Sigma_{\mathcal{N}_e}$. Then the conclusion follows by Lemma 2. ■

Remark 5: In contrast to Theorem 1, the rank condition (22) in Theorem 2 does not imply the matrices (A,B) can be uniquely identified due to the noise on the data. Instead, there exists a set of systems that could have generated the data under the noise models (26). Hence, the results of Theorem 2 are robust data-driven stability conditions.

Remark 6: If $R=\mathbf{0}$ in (26), the collected data $\{u(t)\}_{t=0}^{T-1}$ and $\{x(t)\}_{t=0}^{T}$ are noiseless and $\mathcal{N}_e$ is simplified to $\mathcal{N}_e=\{\mathbf{0}\}$. In this case, LMI (29) in Theorem 2 and LMI (23) in Theorem 1 are equivalent in terms of feasibility. Indeed, when $R=\mathbf{0}$, LMI (29) is equivalent to

$$\begin{bmatrix}Z_1(P)+Z_2(\lambda) & *\\ -\begin{bmatrix}-P&\mathbf{0}\\ \mathbf{0}&P\end{bmatrix}\begin{bmatrix}X_-&\mathbf{0}\\ X_+&\mathbf{0}\end{bmatrix} & \begin{bmatrix}-P-\gamma I_n&\mathbf{0}\\ \mathbf{0}&P-\gamma Q\end{bmatrix}\end{bmatrix}\prec 0. \tag{31}$$

If LMI (31) is feasible, LMI (23) is also feasible for the same P and λ, since $Z_1(P)+Z_2(\lambda)$ is the upper left diagonal block of a negative definite matrix. Conversely, if LMI (23) is feasible, there always exists a large enough $\gamma>0$ such that $\begin{bmatrix}-P-\gamma I&\mathbf{0}\\ \mathbf{0}&P-\gamma Q\end{bmatrix}\prec 0$; therefore, LMI (31) holds by the Schur Complement, for the same P and λ. Thus, Theorem 2 can be considered as a generalization of Theorem 1.

Remark 7: In [18], the following noise model is also considered: $\mathcal{N}_2\triangleq\left\{E\,\middle|\,\begin{bmatrix}I\\ E\end{bmatrix}^\top\begin{bmatrix}\Theta_{11}&\Theta_{12}\\ \Theta_{12}^\top&\Theta_{22}\end{bmatrix}\begin{bmatrix}I\\ E\end{bmatrix}\succeq 0\right\}$, where $\Theta_{11}=\Theta_{11}^\top,\Theta_{12}$, and $\Theta_{22}=\Theta_{22}^\top$ are known matrices; it is also shown therein that the noise models $\mathcal{N}_1$ (see Remark 4) and $\mathcal{N}_2$ are equivalent under certain assumptions.

Theorem 2 can be extended to the noise model $\mathcal{N}_e'=\{E\in\mathbb{R}^{n\times T}\mid E^\top QE\prec R\}$ where $Q\in\mathbb{S}_{\succ 0}^n$ and $R\in\mathbb{S}_{\succeq 0}^T$ are given. Note that this noise model differs from the one in (26) due to the different transpositions of the noise matrix E. Note also that $\mathcal{N}_e'$ is a special case of $\mathcal{N}_2$ with $\Phi_{11}=R,\Phi_{12}=\mathbf{0}$, and $\Phi_{22}=-Q$. If Q^{-1} and R^{-1} in (29) are replaced with R and Q, respectively, with all other conditions in Theorem 2 unchanged, then the conclusion of Theorem 2 still holds with respect to $\mathcal{N}_e'$. The proof follows directly from that of Theorem 2.

Remark 8: The LMI (29) has size $(3n+n_\phi)\times(3n+n_\phi)$. Since the number of neurons n_ϕ typically far exceeds the system dimension n, the computational complexities of the LMIs (23) and (29) are both dominated by the size of the NN controller rather than the system dimension.

IV. Simulation

Consider the linearized inverted pendulum model in [13]:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{\mu}{m\ell^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix} u, \tag{32}$$

where the state $x = [\theta\ \dot{\theta}]^\top \in \mathcal{X} \subset \mathbb{R}^2$ represents the angular position θ and angular velocity $\dot{\theta}$, $u \in \mathbb{R}$ is the control input, the mass is $m = 0.15$ kg, the friction coefficient is $\mu = 0.05$ N·m·s/rad, and the state constraint set is given by $\mathcal{X} = [-2.5, 2.5] \times [-6, 6]$. The continuous-time dynamics is discretized with the sampling time $\Delta t = 0.02$ seconds. The controller u is parameterized as the same FNN controller π in [13], which was obtained via a reinforcement learning process. Specifically, the controller is a two-layer FNN with $n_1 = n_2 = 32$ and $\mathtt{tanh}$ activation functions; an additional layer with an identity weight matrix is appended without altering the input-output mapping of the FNN. Therefore, Assumption 1 is satisfied since $\pi(\mathbf{0}) = \mathbf{0}$. The control input is saturated by $u = sat_{\underline{u}}^{\bar{u}}\pi(\boldsymbol{x})$ with $\bar{u} = -\underline{u} = 0.7$ N·m. The noise model follows (26) with $Q = I_3$ and $R = \epsilon I_2$; this corresponds to bounded total energy of the noise sequence.

We assume that the system matrices are unknown, but that a sequence of input-state data of length $T = 3$ is available, namely $\{u(t)\}_{t=0}^{2}$ and $\{x(t)\}_{t=0}^{3}$. We solve the LMIs (17) and (23) in Theorem 1 when the data are generated by (5), and the LMIs (17) and (29) in Theorem 2 when the data are generated by (6) with noise level $\epsilon = 5 \times 10^{-7}$. In both cases, the objective function is $\min \operatorname{trace}(P)$. Fig. 2 shows inner-approximations of the ROAs obtained by solving the two LMIs above. The volume of the ROA with noiseless data is 1.7601, while that with noisy data is 1.4165.

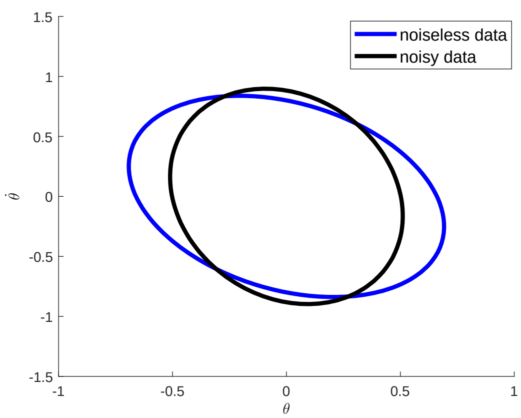

Fig. 2. Inner-approximations of the ROA for the inverted pendulum example using noiseless data (blue) and noisy data (black).

V. Conclusion

We addressed the problem of data-driven stability verification for NNCSs with unknown plant dynamics. LMI conditions were derived that certify stability and yield an estimate of the ROA directly from a finite-length closed-loop trajectory, for both noiseless and noisy data settings. Future work will explore relaxing the data length and richness conditions, extending to more general noise models and broader classes of neural network architectures, and incorporating additional quadratic constraints to reduce conservatism.

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